UNIVERSITY FOR DEVELOPMENT STUDIES

PORTFOLIO OPTIMISATION FOR A FIRM IN GHANA USING LINEAR PROGRAMMING

Bonya Kwame (B.Sc Mathematics)

Thesis submitted to the Department of Mathematics, Faculty of Mathematical Sciences, University for Development Studies in Partial Fulfillment of the Requirements for the Award of Master of Science Degree in Mathematics

2013

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BY

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(UDS/MM/0006/11)

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OCTOBER, 2013

DECLARATION

Student

I hereby declare that this thesis is the result of my original work and that no part of it has been presented for another degree in this University or elsewhere. Related works by others which served as a source of knowledge has been duly referenced and acknowledged.

Candidate's Signature ………………… Date 08 - 11 - 2013

Bonya Kwame

Supervisor

I hereby declare that the preparation and presentation of this thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

Supervisor's Signature ……………………… Date 12/11/13

Dr. S. B. Twum

ABSTRACT

Linear optimisation techniques have been used to find an optimal investment portfolio for a Firm in Ghana. The objective was to find the level of investments in selected portfolios that yielded maximum return for the Firm based on the data supplied. Sensitive analyses were carried out to test the robustness or otherwise of the resulting model to slight changes in the input parameter values. It was also aimed at determining how redundant a constraint was to the solution of the Linear Programming (LP) problem. Simplex Algorithm (implemented on the Quantitative Manager software) was used to solve the resulting LP problems. The portfolio of the Firm consists of a portfolio of investment risks and a portfolio of financial risks. The model is a single objective model that maximises return on the portfolio.

ACKNOWLEDGEMENT

Undertaking this project would not have been possible if the grace, strength and blessings of God have not been on my side. I am very grateful to God.

My sincere thanks goes to my supervisor, Dr. Stephen B. Twum who through his supervisory work assisted me immensely in every stage of the project. I am grateful for the time he dedicated to facilitate my understanding of every concept that was employed in this work and for his numerous effort in the acquisition of relevant data for my analysis.

I also wish to acknowledge his brilliant ideas and display of profound knowledge which have sustained the preparation of the project. I wish to recognize the following people who encouraged me in diverse ways; Dr S.K Amponsah, Mr Boakye Prempeh, Mr Gyabu Kwabena James, Mr Didacus Afegra and Mr Afelibiek Ababu. My immense acknowledgement extends to my family, especially my mum, Madam Poliyoun Abena Danquah and Sister Linda Danquah not forgetting my very good friends Alhaji Braimah Mosi, Alhaji Justice Mahama Ansomah, Mr Muiz Tampson and Mrs Atowie Gifty for their consistent prayers and inspiration. Lastly, my sincere gratitude goes to everyone who has influenced the success of this project in numerous ways.

DEDICATION

I dedicate this thesis to my family and all known faces.

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CHAPTER ONE

INTRODUCTION

1.1 Research Background

The term portfolio refers to any collection of financial assets such as stocks, bonds, and cash. Portfolios may be held by an individual investor and/or managed by financial professionals, hedge funds, banks or other financial institutions. It is a generally accepted principle that a portfolio is designed according to the investor's risk tolerance, time frame and investment objectives. Optimal refers to the best or most favorable among a set of alternatives (Paphristodoulou, 2003). Thus, an optimal portfolio is the portfolio that considers the investors own ''greed'' and/or how risk averse he/she is. Two of the most important concepts in portfolio management are return and risk. This note describes a method for quantifying return, risk, and their tradeoff and for finding optimal portfolios using linear programming. Suppose a portfolio manager is deciding how to allocate funds among several investments alternatives.

In this research linear programming, a mathematical technique used in computer modeling (simulation) to find the best possible solution in allocating limited resources (energy, machines, material, money, personnel, space, time, etc.) to achieve maximum profit or minimum cost, is used to determine the optimal

portfolio for the firm under consideration. However, it is applicable only where all relationships are linear and can accommodate only a limited class of cost functions.

1.2 Problem Statement

The challenge of portfolio optimisation is an important one in investment and finance. It is faced by portfolio holders and managers who are the major decision makers in allocating their resources across different categories. Protection against the risks related with everyday life is indispensable to every human being in order to enjoy some level of peace of mind. The problem is to determine how much of available funds to be allocated to each component of the portfolio to maximised returns on the portfolio.

1.3 Aims and Objectives

The study therefore is concerned with determining optimally the possible levels of investment in selected portfolios of the firm under discussion with emphasis on return maximisation. The specific objectives are to:

i. Model a firm's portfolio selection problem as LP

ii. Determine the optimal portfolios of the firm

iii. Assess the model for stability

iv. Propose an optimum investment for the firm

1.4 Research Questions

Questions needed to answer in this study are:

i What are the areas of investment of the firm?

ii. What are the investment constraints of the firm?

iii. What should be the levels of investments in order to maximise returns?

1.5 Significance of the Study

The study is to determine the levels of investments of the firm for maximum returns. The used of linear programming as a mathematical tool in maximising or minimising a linear objective function subject to linear constraints is extensively used in achieving the firm's objectives. The study would help portfolio managers and holders decide with minimised level of error in allocating their available resources across the different component of investment that make up the portfolio in order to maximise the return on the portfolio. Quantitative Manager, the software package used in solving the resulting LP problem, as the study would show, is not enough in concluding and interpreting its printout. Stakeholders would appreciate the importance of assessing the stability or otherwise of the LP problem of the firm for the best possible conclusion.

1.6 Definition of Terms

Portfolio

The term portfolio refers to any collection of financial assets such as stocks, bonds, and cash. Portfolios may be held by individual investors and/or managed by financial professionals, hedge funds, banks and other financial institutions. It is a generally accepted principle that a portfolio is designed according to the investor's risk tolerance, time frame and investment objectives.

Portfolio selection

It is a range of investments held by an individual or an organisation.

Linear Programming (LP)

It is a mathematical technique for maximizing or minimizing a linear function subject to linear constraints. It begins with the construction of a mathematical model to represent a real life problem.

Optimal portfolio

Optimal refers to the best or most favorable among a lot. Thus, an optimal portfolio is the portfolio that considers the investors own greed and/or how risk averse he/she is and it is subjective.

Return

A profit from an investment.

Slack variable

It is a variable that measures the amount of slack (idle) resources still remaining in stock at any point in time during the production process. Since it is not possible to have negative slack, nonnegative constraints also apply to slack variables.

Risk

A situation involving exposure to danger.

Consider the LP in standard form:

Optimise: $Z = C^T x$

Subject to: $Ax = b$,

$x \geq 0$.

Feasible Set/ Region

Let K be a feasible set of the standard LP above. Then

$K = \{x \in R^n : Ax = b, x \geq 0\}$.

Feasible Solution

x is a feasible solution if $x \in K$ or if x satisfies $Ax = b$ and $x \geq 0$, then x is a feasible solution.

Optimal Solution

Let $C^T x$ be the objective function of a LP to be optimised. $x \in$ K is an optimal solution if for all

$y \in K, C^T x > C^T y$.

Basic feasible solution

Let x be a basic solution of $Ax = b$. If $x \geq 0$, then it is called a basic feasible solution (BFS).

Algorithm

An algorithm is a set of rules or systematic procedure for finding the solution to a problem.

Simplex algorithm

Simplex algorithm is a method or computational procedure for determining basic feasible solutions to a system of equations and testing the solutions for optimality.

Artificial variable

An artificial variable $A_i \geq 0$ is a dummy variable added for the specific purpose of generating an initial basic feasible solution. It has no economic meaning.

CHAPTER TWO

REVIEW OF LITERATURE

2.1 Introduction

Optimisation, or constrained optimisation, or mathematical programming, is a mathematical procedure for determining optimal allocation of scarce resources. Optimisation, and its most popular special form, Linear Programming (LP), has found practical application in almost all facets of business, from advertising to production planning. Transportation and aggregate production planning problems are the most typical objects of LP analysis (Bazaraa etal, 2002). The petroleum industry was an early intensive user of LP for solving fuel blending problems. Optimisation originated in the 1940s, when George Dantzig used mathematical techniques for generating "programs" (training timetables and schedules) for military application (Young, 1998). Since then, his "linear programming" techniques and their descendents were applied to a wide variety of problems, from the scheduling of production facilities, to yield management in airlines. Today, optimisation comprises a wide variety of techniques from Operations Research, artificial intelligence and computer science, and is used to improve business processes in practically all industries (Bacon, C. 2008).

"Programming" in Mathematical Programming is of a different flavor than the "programming" in Computer Programming. In the former case, it means to plan and organise (as in "Get with the program!"). In the latter case, it means to write instructions for performing calculations. Although aptitude in one suggests aptitude in the other, training in the one kind of programming has very little direct relevance to the other (Konno and Yamazaki, 1991). For most optimisation problems, one can think of there being two important classes of objects. The first of these is limited resources, such as land, plant capacity, and sales force size. The second is activities, such as "produce low carbon steel," "produce stainless steel," and "produce high carbon steel." Each activity consumes or possibly contributes additional amounts of the resources (Brinson, G., and Beebower, G. 1991), The problem is to determine the best combination of activity levels that does not use more resources than are actually available.

2.2 Portfolio Optimisation Models

The problem of portfolio optimisation has been one of the standard and most important problems in the fields of financial research and investment (Brinson, etal 1986). In the nineteen-fifties Harry Markowitz introduced to the world an elegant way of managing risk in financial markets known as portfolio theory (Markowitz, 1959). Prior to Markowitz's work, investors focused on assessing the risks and rewards of individual securities in constructing their portfolios. Standard investment advice was to identify those securities that offered the best opportunities for gain with the least risk and then construct a portfolio from these (Davis, M. and Norman, A.1990). Following this advice, an investor might conclude that bank stocks all offered good risk-reward characteristics and compile a portfolio entirely from these. The Markowitz model is a single-period model, where an investor forms a portfolio at the beginning of the period(Konno and Yamazak, 1991). The investor's objective is to maximise the portfolio's expected return subject to an acceptable level of risk or minimise risk subject to an acceptable expected return (Dhaene etal, 2005).

William Sharpe later extended such a theory by introducing the Capital Asset Pricing Model (CAPM) (Sharpe, 1964). Portfolio theory is based upon the empirical observation that expected return and return variance are positive serial correlated (Elton etal, 1995). Such an empirical fact is very convenient for a portfolio investor that is trying to maximise risk adjusted return (Sharpe, 1971). William Sharpe, who among others tried to simplify the process of data inputs, data tabulation, and reaching a solution, developed a simplified variant of the Markowitz model that reduces substantially its data and computational requirements (Harrington, D. R. 1987). William F. Sharpe's pioneering achievement in this field was contained in his essay entitled to Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk (Sharpe, 1964). As per Sharpe model or Portfolio optimisation model, the linearity of security should be found.

2.3 Asset Allocation and Investment Management Process

Setting investment policy guidelines begins with the asset allocation decision, how the funds to be invested should be distributed among the major classes of assets. An asset allocation model is used to provide guidance in making this decision (Włodzimierz, 2000). While there are many asset allocation models proposed, the critical input in all of them is the expected return for an asset class. The expected return for an asset class is estimated using the Simplex Algorithm, an application of linear programming. The process is explained under methodology in chapter three. Selecting a portfolio strategy, clients can request for a money manager for a particular asset class to pursue an active or passive strategy (Maginn etal, 1990) . An active portfolio strategy uses available information and forecasting techniques to seek a better performance than a portfolio that is simply diversified broadly (Wayne, 2003). A passive portfolio strategy involves minimal expectation input, and instead relies on diversification to match the performance of some market index. There are also hybrid strategies. Whether clients select an active or passive strategy depends on their belief that the market is efficient for an asset class (Rubinstein, M. 2006).

Portfolio manager minimises the tracking error versus a benchmark but at the same time in the constraint set bounds from above the volatility of the portfolio return (Kostreva and Ogryczak, 2000). In this way, if the benchmark is volatile, the portfolio manager requires that the portfolio's volatility is bounded.

Consequently, most studies have focused on the pricing efficiency of the equity markets. Pricing efficiency refers to a market where prices at all times fully reflect all available information that is relevant to the valuation of securities (Marcus, 2011).

When a market is price-efficient, strategies pursued to outperform a broad-based market index will not consistently produce superior returns after adjusting for both risk and transactions costs (Saunders etal, 2003) . In this study Linear Programming is used to determine optimally the possible portfolio mix for the firm in order to maximise the expected return on the portfolio. Linear Programming is used due to its flexibility in operating and the ease in interpreting the solution printout as embedded in the Quantitative Manager.

CHAPTER THREE

METHODOLOGY

3.1 Data Collection

The study used only secondary data supplied by the firm. The data gives the areas of investments identified by the firm and their corresponding risk associated with each investment area. It also spelt out the conditions governing the distribution of the available funds for the investment before Linear Programming used in achieving the objectives of the firm.

3.2 Overview of Linear Programming (LP)

LP is concerned with the theory and methods of maximizing or minimizing a linear function subject to linear constraints. In management terms, it could be said to be finding the best or efficient way to utilize scarce resources. The most difficult aspect of solving a constrained optimisation problem by LP is to formulate or state the problem in a linear programming format or framework (Catherine S., 2008). Depending on the number of decision variables, LP problems are easily solved graphically or by the use of computers (Thomas, 1998). It is important, however, to know the process by which even the most complex LP problems are formulated and solved and how the results are interpreted. In this study we determine using LP the optimal return in a portfolio mix subject to constraints based on a firm's policies and availability of funds.

LP begins with the construction of mathematical model to represent a real problem. The general LP model is of the form

Optimise: $f(x)$

Subject to: $g_i(x) \leq b_i, \quad 1 \leq i \leq p$

$g_i(x) = b_i, \quad p+1 \leq i \leq k$

$g_i(x) \geq b_i, \quad k+1 \leq i \leq n,$

where $f(x)$ is a linear function of a vector variable, $x = (x_1, x_2, \ldots, x_n)^T$ and $g_i(x)$ $(1 \leq i \leq m)$ over constraint functions of x. The variable x_j, $j = 1, 2, \ldots, n$ are activity levels associated with the decision making problem of which an LP model represents and which is under the control of the decision maker. linear they are precisely defined in the form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n = \sum_{j=1}^{n} c_j x_j \quad \ldots\ldots\ldots\ldots\ldots (3.1)$$

$$g_i(x) = a_{i1}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots a_{in}x_n = \sum_{j=1}^{n} a_{ij}x_j \quad \ldots\ldots(3.2)$$

The LP in (1) and (2) can thus be written in the following form:

Optimise:

$$\sum_{j=1}^{n} c_j x_j$$

Subject to:

$$\sum_{j=1}^{n} a_{ij}x_j,\ \ 1 \le i \le p$$

$$\sum_{j=1}^{n} a_{ij}x_j,\ \ p+1 \le i \le k$$

$$\sum_{j=1}^{m} a_{ij}x_j,\ k+1 \le i \le m$$

$$x_j \ge 0,\ 1 \le j \le n.$$

The quantities a_{ij} and c_j are called Technological and Cost Coefficients respectively. These together with the RHS b_i constitute the main parameters of the models.

3.3 The General, Canonical and Standard form LP

An LP is said to be in a general form if it is of the form

Optimise:

$$\sum_{j=1}^{n} c_j\, x_j$$

Subject to:

$$\sum_{j=1}^{n} a_{ij}x_j,\ \ 1 \le i \le p$$

$$\sum_{j=1}^{n} a_{ij}x_j,\ \ p+1 \le i \le k$$

$$\sum_{j=1}^{m} a_{ij}x_j,\ k+1 \le i \le m;\ x_j \ge 0,\ 1 \le j \le n.$$

When the LP is of the form

Optimise:

$$\sum_{j=1}^{n} c_j\, x_j$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \text{ for all } i, \;\; x_j \geq 0, \text{ for all } j;$$

then it is called the canonical form LP.

When the LP is of the form

Optimise:

$$\sum_{j=1}^{n} c_j\, x_j$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \text{ for all } i,$$

$$x_j \geq 0, \text{ for all } j;$$

then the LP is said to be in standard form.

All LP problems can be transformed into the standard form. The standard form LP is very important since LP algorithms work under equality conditions only. The standard and canonical form LPs may be expressed in the matrix form in the following forms respectively:

Optimise: $C^T x$

Subject to: $Ax \leq b,$

$x \geq 0$ and

Optimise: $C^T x$

Subject to: $Ax \leq b,$

$x \geq 0,$

where $C = (c_1, c_2, \ldots, c_n)^T,$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} \ldots\ldots\ldots & a_{mn} \end{pmatrix}, b = (b_1, b_2, b_3, \ldots, b_m)^T.$$

3.4 Conversion of LP into Standard form

An inequality constraint of the form

$$a_{i1}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots a_{in}x_n \leq b_i, \quad 1 \leq i \leq m$$

can be converted to equality form by the introduction of slack variables $s_i \geq 0$ into the inequality. This gives

$$a_{i1}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots a_{in}x_n + s_i = b_i,$$

where s_i, for all i is a variable.

An inequality constraint of the form

$$a_{i1}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots a_{in}x_n \geq b_i,$$

for all i may be converted into an equality of the form

$$a_{i1}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots a_{in}x_n - w_i = b_i,$$

for all i where $w_i \geq 0$ $(1 \leq i \leq m)$. w_i is called a *surplus variable.*

NOTE

LP constraints are limited to only three possible relational operators: $=$, $\leq$ and $\geq$. Strict inequalities ($<$, $>$) are not allowed in LP.

3.5 Assumptions of Linear Programming (LP)

Linear programming is a mathematical technique for solving constrained maximisation or minimisation problems where the constraints and the objective function being optimised are linear (i.e., can be represented by straight lines). Firms and other organisations face many constraints in achieving their goals of profit maximisation, cost minimisation, or other objectives. A Linear Programming model has four main inherent assumptions namely; Proportionality, Additivity, Divisibility and Certainty.

3.5.1 Proportionality

The contribution of any variable to the objective function or constraints is proportional to that value of the variable. This implies no discounts or economies to scale. Thus an activity level x_k $(1 \leq k \leq n)$ contributes $c_k x_k$ and $a_{ik} x_k$ respectively to the objective and the constraint functions. The contribution to both is proportional to the activity level. However there may be cases where this assumption may not hold precisely but can be used as an approximation.

3.5.2 Additivity

The contribution of any variable to the objective function or constraints is independent of the values of the other variables. It supposes that there are no interactions within or between any of the activity levels so that terms such as x_k^2 or $x_k x_{k+1}$ where $1 \leq k \leq n$ are non-existent in either the objective function or the constraint functions. The assumption implies that given activity levels x_k and x_p, say, their total contribution to both the objective function and constraint function is the sum of their individual contributions.

3.5.3 Divisibility

Decision variables can be fractions. It requires that an activity level may be divided into any fractional levels so that non- integer values are admissible. In cases where integer values are strictly required the values obtained after solving LP model may be rounded to the nearest integer. This practice may not always be useful, since such a modification can seriously result in a sub-optimal solution where the Linear Programming model is sensitive.

3.5.4 Certainty

This assumption is also called the deterministic assumption. This means that all parameters (all coefficients in the objective function and the constraints) are known with certainty. Realistically, however, coefficients and parameters are often the result of guess-work and approximation. The effect of changing these numbers can be determined with sensitivity analysis.

3.6 Formulation of the Profit Maximisation Linear Programming Problem

Most firms invest in more than one investment opportunities, and a crucial question to which they seek an answer is how much of its capital at hand should be channeled into each investment opportunity in order to maximise profits. Usually, firms also face many constraints on the availability of the inputs they use in their investment activities and management policies. The problem is then to determine the level of investments that maximise the firm's total profit subject to the input constraints it faces. The formulation is done as follows:

(i) The objective function to be optimised is express as a linear equation.

(ii) The firm's constraints are also expressed as linear inequalities. (The reason is that the firm can often use up to, but not more than, specified amounts to be invested, or the firm must meet some minimum requirement).

(iii) In addition, there are nonnegative constraints on the solution to indicate that the firm cannot produce a negative output or use a negative quantity of any input.

(iv) Graph the inequality constraints, and define the feasible region. This is possible when the decision variables are only two; in which case we would be dealing with two dimensional space.

(v) Graph the objective function as a series of isoprofit (i.e., equal profit) lines, one for each level of profit respectively.(for only 2-dimenssional space)

(vi) Find the optimal solution (i.e., the values of the decision variables) at the extreme point or corner of the feasible region that touches the highest isoprofit line. This represents the optimal solution to the problem subject to the constraints faced.

3.7 The Simplex Method

The Simplex method is a widely used algorithm for solving large scale LP problems (in particular where ones geometric intuition falters and one has to rely on algebraic means in order to proceed) and all LPs in general. The graphical approach solves LP problems by identifying the extreme point of the feasible set and testing the objective function value at the extreme points. The one which yields the best value for the objective function is selected as the optimal solution.

The Simplex algorithm thus does the same thing using purely algebraic means. The algebraic means is necessary in higher dimensional (ie $n \geq 3$) due to our inability to perceive the feasible region or the objective function geometrically. In higher dimension the objective and constraint functions are not line segments but hyper-planes (a geometrical concept of a plane in higher dimension) and the feasible region not a plane polygon but a simplex which is a region bounded by hyper-planes.

3.8.1 Simplex Method with 'less-than-equal-to' ($\leq$) constraints

The LP problem, with 'less-than-equal-to' ($\leq$) constraints, is transformed to its standard form in the following way.

i. Putting the LP in standard form

ii. Finding an initial basic feasible solution (IBFS)

iii. Checking whether the IBFS is optimal

iv. If the IBFS is not optimal, find a new BFS with a better objective function value

v. Repeating steps 3 and 4 until there is no better value of the objective function, indicating that the current value was optimal, or until it is clear that there is no optimal solution.

3.8.2 Simplex Method with 'Greater-Than-Equal-to' ($\geq$) and Equality (=) Constraints

'Greater-than-equal-to' ($\geq$) and 'equality' (=) constraints are also possible in LP problems. In such cases, a modified approach of the Simplex algorithm is followed. The LP problem, with 'greater-than-equal-to' ($\geq$) and equality (=) constraints, is transformed to its standard form in the following way.

i. One 'artificial variable' is added to each of the 'greater-than-equal-to' ($\geq$) and equality (=) constraints to ensure an initial basic feasible solution.

ii. Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient M for maximization (minimization)

iii. Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.

iv. If there are more than one artificial variable, step 3 is repeated for all the artificial variables one by one.

3.8 Minimization Versus Maximization Problems

As discussed earlier, standard form LP problems consist of a maximizing objective function. Simplex method is described based on the standard form of LP problems, i.e. objective function is of maximization type. However, if the objective function is of minimization type, simplex method may still be applied with a small modification. The required modification can be done in either of following two ways.

i. The objective function is multiplied by -1 so as to keep the problem identical and 'minimization' problem becomes 'maximization'. This is because of the fact that minimizing a function is equivalent to the maximization of its negative.

ii. While selecting the entering non-basic variable, the variable having the maximum coefficient among all the cost coefficients is to be entered. In such cases, optimal solution would be determined from the table having all the cost coefficients as nonpositive (≤ 0). Still one difficulty remains in the minimization problem. Generally the minimization problems consist of constraints with 'greater-than-equal-to' ($\geq$) sign. For example, minimise the price (to compete in the market); however, the profit should cross a minimum threshold. Whenever the goal is to minimise some objective, lower bounded requirements play the leading role. Constraints with 'greater-than-equal-to'($\geq$) sign are obvious in practical situations.

3.9 The Solution Algorithm

The Simplex method is an iterative search algorithm for large LP problems, starting from the initial ("origin", all $x = 0$) and moving toward adjacent "corner" points at the direction in which improvement on objective function value is maximised. If one "corner" point solution is better than all adjacent "corner" point solution, it is "optimal". The Simplex algorithm examines a finite number of basic feasible solutions (BFS) for optimality. For large size problems where the number of BFS's can be very large, the algorithm subset of the BFS's to examine for optimality. Even so, for very large size problems, the subset of BFS's to examine could still be large or very large. Software packages of the Simplex algorithm are bound today and can be implemented on a computer.

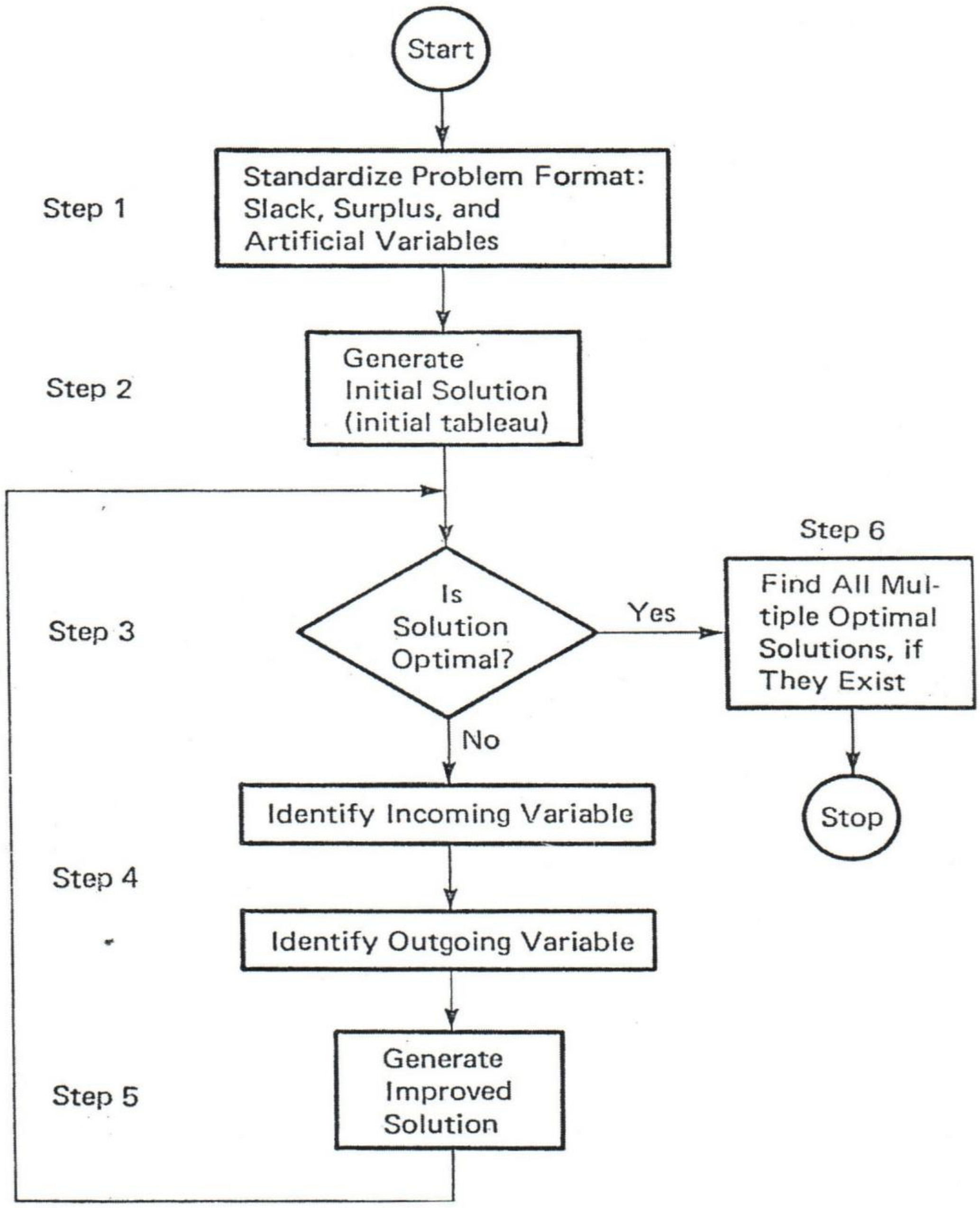

Figure 3.1 Schematic presentation of the Simplex method

NOTE

All '' $\geq$ '' constraints are automatically converted by LP software into their standard form (equations) by creating and subtracting surplus variables (w_i) *whilst all ''$\leq$ '' constraints are also converted into their standard form (equations) by creating and adding slack variables* (s_i).

3.10 Sensitivity Analysis

It is a technique used to determine how different values of an independent variable will impact a particular dependent variable under a given set of assumptions. This technique is used within specific boundaries that will depend on one or more input variables, such as the effect that changes in interest rates will have on a bond's price. Sensitivity analysis is a way to predict the outcome of a decision if a situation turns out to be different compared to the key prediction(s). Sensitivity Analysis concerns the impacts on optimal solution if one or more given model parameters are changed due to market/ supply/ capacity/ technology/... changes. Sensitivity analysis is very useful when attempting to determine the impact the actual outcome of a particular variable will have if it differs from what was previously assumed.

By creating a given set of scenarios, the analyst can determine how changes in one variable(s) will impact the target variable. For example, an analyst might create a financial model that will value a company's equity (the dependent variable) given the amount of earnings per share (an independent variable) the company reports at the end of the year and the company's price-to-earnings multiple (another independent variable) at that time. The analyst can create a table of predicted price-to-earnings multiples and a corresponding value of the company's equity based on different values for each of the independent variables.

3.10.1 Importance of Sensitivity Analysis

Sensitivity analysis is very useful when attempting to determine the impact the actual outcome of a particular variable will have if it differs from what was previously assumed. By creating a given set of scenarios, the analyst can determine how changes in one variable(s) will impact the target variable. Different LP packages have different formats for input/output but the same information as discussed above is still obtained. Essentially the interpretation of LP output is something that comes with practice. Much of the information obtainable (as discussed above) as a by-product of the solution of the LP problem can be useful to management in estimating the effect of changes (e.g. changes in costs, production capacities, profit margins, etc) without going to the hassle/expense of resolving the LP. This sensitivity information gives us a measure of how robust the solution is i.e. how sensitive it is to changes in input data. Conducting a sensitivity analysis is beneficial in several ways. Not only can one make better and more informed decisions by changing assumptions and observing, or estimating the results, one is also better able to predict the outcome of his decisions. For example, if sensitivity analysis is conducted before deciding to increase prices, the decision is less risky than if one did not go through this exercise. In addition to proving with a glimpse into the future, sensitivity analysis leads to faster decisions.

At a general level, Sensitivity analysis refers to a statistical concept in which one variable is changed and other variables are held constant. This allows modelers, including developers and investors, to ascertain and review the effects of each variable being changed. Usually the analysis is conducted to asset the net present value (or other key performance indicators) of the property and work out which are the best scenarios for buying or selling investment property.

3.11 Data Analysis Tools

Software Packages exist for the implementation of the Simplex method in a computer towards solving linear optimisation problems. These Software Packages vary by their degree of flexibility, user friendliness, ability to handle large scale problem, and computational time to output the result. Some of these tools are; CPLEX, OSL, OBI, Microsoft Excel Solver and Quantitative Manager (QM). Thus, at the end of formulating the model, any of these software packages can be used to solve the LP problem that would be formulated from the data gathered. Any one of these gives the same output of analysis. However the Quantitative Manager was used for the analysis due to its simplicity and accuracy.

CHAPTER FOUR

DATA PRESENTATION AND RESULTS

4.1 Description of the Problem

The Firm under consideration has at its disposal, fifteen million Ghana Cedis (Gh¢ 15,000,000.00) for the period considered to invest among seven (7) investment options with the aim of maximising returns. After identifying the preferred areas of investment, the distribution of available funds among each investment area in order to derive the maximum return under acceptable level of risk becomes a major headache. The decision for distribution of funds to the various investments should not be done arbitrarily, since there cannot be any guarantee of achieving the goal of maximum return. A scientific approaching to the problem therefore is the best way forward.

The Firm, according to their own records, does not adopt any scientific procedure for its investment decision making and thus is not able to tell whether it has been making the desired return on its investments or not over the past years. The decision to go scientific for the period under consideration was informed by the changing trends in the world of business.

Table 4.1 shows the summary of investment options identified by the firm to invest the available funds to achieve its aim. These are Fixed Deposit, Treasury Bills, Cocoa Beans, Mortgage Securities, Construction Loans, Cashew Nuts, Apex Certificate of Deposit (ACOD).

Table 4.1: Investment Areas, Interest Rates and Risk Scores

Investment	Interest Rate (%)	Risk Score (%)
Fixed Deposit	10.0	1.7
Treasury Bills	20.5	2.5
Cocoa Beans	20.0	1.5
Mortgage Securities	8.0	1.9
Construction Loans	28.5	2.9
Cashew Nuts	18.5	1.5
ACOD	9.5	2.2

4.2 Calculation of Investment Risk

The level of risk (i.e. risk score) associated with each investment is obtained by taking the average return (in percentage) over at least the past five (5) years. The mean (average) of the returns on the investment for the period considered is calculated and the sum of the squares of the deviations is divided by the number of years considered minus one and the square root of the result taken.

4.3 Constraints of the Problem

There are six (6) constraints associated with the problem; these are identified below as follows:

(i) Invest up to Gh¢ 15,000,000.00 in the entire investment options

The Firm is constrained not to go beyond investing Gh¢ 15,000,000.00 in the entire seven investment areas identified. However investing less that Gh¢ 15,000,000.00 is allowed. Thus at the end of distributing the available funds among the seven investments, the total the distribution should not exceed Gh¢ 15,000,000.00.

(ii) Not more than 20% of total investments into any one investment area

Investment in any one of the seven investment areas should not exceed 20% of the total investment. Thus the quantum of money that is suppose to go into any one of the seven investments can be less than 20% of the total money for entire investments (ie ≤ 20% of Gh¢ 15,000,000.00).

(iii) At least 25% of total investments into Deposits

The Firm is not allowed to commit less than 25% of the total amount of money into Deposits (ie. Fixed Deposit and ACOD). Hence the Firm is to commit not less than 25% of Gh¢ 15,000,000.00 into Fixed Deposit and ACOD.

(iv) At least 30% of total investments into Cash Crops

The Firm is not allowed to commit less than 30% of the total amount of money into Cash Crops (ie. Cocoa Beans and Cashew Nuts). Hence the Firm is to commit not less than 30% of Gh¢ 15,000,000.00 into Cocoa Beans and Cashew Nuts.

(v) At least 45% of total investments into Treasury Bills and Construction Loans

The Firm is not allowed to commit less than 45% of the total amount of money into Treasury Bills and Construction Loans. Hence the Firm is to commit not less than 45% of Gh¢ 15,000,000.00 into Treasury Bills and Construction Loans.

(vi) Limit overall risk to not more than 2.0%

Portfolio risk which is calculated using the weighted average should be equal or less than 2.0%.

Short term Treasury Bills (six months Treasury Bills) are risk-free while that of long term (more than six months) are not (Bank of Ghana). The risk set in when prime rates are reviewed upwards.

4.4 Modeling the Problem as LP

This section discusses the model formulation process which principally concerns defining the decision variables and formulating the objective and constraint functions. The following present the notation and definition of the decision variables of the problem.

x_1 : Gh¢ to invest in Fixed Deposit

x_2 : Gh¢ to invest in Treasury Bills

x_3 : Gh¢ to invest in Cocoa Beans

x_4 : Gh¢ to invest in Mortgage Securities

x_5 : Gh¢ to invest in Construction Loans

x_6 : Gh¢ to invest in Cashew Nuts

x_7 : Gh¢ to invest in Apex Certificate of Deposit (ACOD)

The objective function in terms of the decision variables is as follows:

Maximise:

$$Z = 0.1x_1 + 0.205x_2 + 0.2x_3 + 0.08x_4 + 0.285x_5 + 0.185x_6 + 0.095x_7$$

The individual constraints identified are expressed quantitatively as follows:

Investment of Gh¢ **15,000,000.00**

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 15{,}000{,}000$$

Not more than 20% of total investment in a particular investment

$$x_1 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$x_2 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$x_3 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$x_4 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$x_5 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$x_6 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$x_7 \leq 0.2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

The above are equivalently expressed as:

$$0.8x_1 - 0.2\,x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 - 0.2x_6 - 0.2x_7 \leq 0$$

$$0.2x_1 - 0.8\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 - 0.8x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 - 0.8\,x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 - 0.8x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 - 0.8x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 - 0.8x_7 \geq 0$$

At least 25% of total investment into Deposits

$$\frac{x_1 + x_7}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \geq \frac{25}{100}$$

$$\equiv$$

$$\frac{x_1 + x_7}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \geq 0.25$$

$$\equiv$$

$$x_1 + x_7 \geq 0.25(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

which is equivalent to

$$0.75x_1 - 0.25x_2 - 0.25x_3 - 0.25x_4 - 0.25x_5 - 0.25x_6 + 0.75x_7 \geq 0$$

At least 30% of total investment into Cash Crops

$$\frac{x_3 + x_6}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \geq \frac{30}{100}$$

$$\equiv$$

$$x_3 + x_6 \geq 0.3(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

which is equivalent to

$$0.3x_1 + 0.3x_2 - 0.7x_3 + 0.3x_4 + 0.3x_5 - 0.7x_6 + 0.3x_7 \leq 0$$

At least 45% total investment into Treasury Bills and Construction Loans

$$\frac{x_2 + x_5}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \geq \frac{45}{100}$$

$$\equiv$$

$$x_2 + x_5 \geq 0.45(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

which is equivalent to

$$0.45x_1 - 0.55x_2 + 0.45x_3 + 0.45x_4 - 0.55x_5 + 0.45x_6 + 0.45x_7 \leq 0$$

***Overall risk limited to not more than* 2.0**

(*Use a weighted average to calculate portfolio risk*)

$$\frac{1.7x_1+2.5x_2+1.5x_3+1.9x_4+2.9x_5+1.5x_6+2.2x_7}{x_1+x_2+x_3+x_4+x_5+x_6+x_7} \leq 2.0$$

OR

$$1.7x_1 + 2.5x_2 + 1.5x_3 + 1.9x_4 + 2.9x_5 + 1.5x_6 + 2.2x_7 \leq$$

$$2.0(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

which is equivalent to

$$0.3x_1 - 0.5x_2 + 0.5x_3 + 0.1x_4 - 0.9x_5 + 0.5x_6 - 0.2x_7 \geq 0$$

Non-negativity of decision variables

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

The resulting LP model is therefore as follows:

Maximise:

$$Z = 0.1x_1 + 0.205x_2 + 0.2x_3 + 0.08x_4 + 0.285x_5 + 0.185x_6 + 0.095x_7$$

Subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 15\,000{,}000$$

$$0.8x_1 - 0.2\,x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 + 0.2x_6 + 0.2x_7 \leq 0$$

$$0.2x_1 - 0.8\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 - 0.8x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 - 0.8\,x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 - 0.8x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 - 0.8x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 - 0.8x_7 \geq 0$$

$$0.75x_1 - 0.25x_2 - 0.25x_3 - 0.25x_4 - 0.25x_5 - 0.25x_6 + 0.75x_7 \geq 0$$

$$0.3x_1 + 0.3x_2 - 0.7x_3 + 0.3x_4 + 0.3x_5 - 0.7x_6 + 0.3x_7 \leq 0$$

$$0.45x_1 - 0.55x_2 + 0.45x_3 + 0.45x_4 - 0.55x_5 + 0.45x_6 + 0.45x_7 \leq 0$$

$$0.3x_1 - 0.5x_2 + 0.5x_3 + 0.1x_4 - 0.9x_5 + 0.5x_6 - 0.2x_7 \geq 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.1)$$

Solution

The Quantitative Manager printout below indicates the optimal investment in a portfolio mix for the firm in order to maximise returns subject to the firm's investment policies, constraints and available funds is as follows:

x_1 (Fixed Deposit) = Gh¢ 2,674,300.00

x_2 (Treasury Bills) = Gh¢ 2,554,800.00

x_3 (Cocoa Beans) = Gh¢ 1,974,800.00

x_4 (Mortgage Securities) = Gh¢ 813,480.00

x_5 (Construction Loans) = Gh¢ 2,407,320.00

x_6 (Cashew Nuts) = Gh¢ 2,674,800.00

x_7 (Apex Certificate of Deposit) = Gh¢ 1,900,500.00

The maximum return = Gh¢ 2,553,096.55

Thus each figure associated with each investment is the amount to commit to that investment in order to make a maximum return on the entire investment and the maximum return is **Gh¢ 2,553,096.55.**

4.5 Sensitivity Analysis

Sensitive analysis in finance is the part of capital budgeting decisions. The stability or robustness of the model is tested by slight changes in the technological coefficients and also to determine the redundancy or otherwise of one of the constraints. This would help make better recommendations and reduce errors in making decisions. Interest rate on each investment is reduced by 5% and the resulting LP problem solved and the solution to the real LP problem. We now analyse the solution to the LP problem when interest rates are now increased by 5% when compared to the original LP problem. The redundancy of a constraint is also put into test and the solution compared to the original LP problem. Each scenario is discussed as follows:

(i) Reducing interest rates by 5% while maintaining risk scores and Firm's constraint

Interest rate on each investment is reduced by 5% as follows:

Fixed Deposit: $\frac{95}{100} \times 10\% = 9.5\%$

Treasury Bills: $\frac{95}{100} \times 20.5\% = 19.475\%$

Cocoa Beans: $\frac{95}{100} \times 20\% = 19\%$

Mortgage Securities: $\frac{95}{100} \times 8\% = 7.6\%$

Construction Loans: $\frac{95}{100} \times 28.5\% = 27.075\%$

Cashew Nuts: $\frac{95}{100} \times 18.5\% = 17.575\%$

ACOD: $\frac{95}{100} \times 9.5\% = 9.025\%$

Table 4.2: Interest rates down by 5%

Investment	Interest Rate (%)	Risk Score (%)
Fixed Deposit	9.5	1.7
Treasury Bills	19.475	2.5
Cocoa Beans	19	1.5
Mortgage Securities	7.6	1.9
Construction Loans	27.075	2.9
Cashew Nuts	17.575	1.5
ACOD	9.025	2.2

As interest rates down by 5% only the technological coefficients of the decision variables changes while cost coefficients and the right hand side remains constant.

Thus the objective function becomes:

$$Z = 0.095x_1 + 0.19475x_2 + 0.19x_3 + 0.076x_4 + 0.27075x_5 + 0.17575x_6 + 0.09025x_7$$

Hence, the problem is formulated as LP problem as follows:

***Maximise*:**

$$Z = 0.095x_1 + 0.19475x_2 + 0.19x_3 + 0.076x_4 + 0.27075x_5 + 0.17575x_6 + 0.09025x_7$$

***Subject to constraints* (4.1).**

Presentation of the solution

The technological coefficients of the revised function values and the original constraint function values are imputed into the computer and the Quantitative Manager software used to determine the output. The output was summarized as follows:

x_1 (Fixed Deposit) = Gh¢ 2,399,800.00

x_2 (Treasury Bills) = Gh¢ 1,874,300.00

x_3 (Cocoa Beans) = Gh¢ 2,129,800.00

x_4 (Mortgage Securities) = Gh¢ 2,207,320.00

x_5 (Construction Loans) = Gh¢ 1,013,480.00

x_6 (Cashew Nuts) = Gh¢ 2,674,800.00

x_7 (ACO D) = Gh¢ 2,7005,00.00

The maximum return = **Gh¢ 2,170,132.068**

Thus, each figure associated with each investment is the amount to commit to the investment in order to make a maximum return on the entire investment and the maximum return is **Gh¢ 2,170,132.068.** It is observed from the analysis above that as interest rates goes down by 5%, investment in Treasury Bills and investment in Construction Loans depreciate significantly. The expected return also experiences a reduction in value by 15% (ie, from **Gh¢ 2,553,096.55** to **Gh¢ 2,170,132.068**)

(*b*) Increasing interest rates by 5% while maintaining risk scores and constraints

Interest rate on each investment is increased by 5% as follows:

Fixed Deposit: $\frac{105}{100} \times 10\% = 10.5\%$

Treasury Bills: $\frac{105}{100} \times 20.5\% = 21.525\%$

Cocoa Beans: $\frac{105}{100} \times 20\% = 21\%$

Mortgage Securities: $\frac{105}{100} \times 8\% = 8.4\%$

Construction Loans: $\frac{105}{100} \times 28.5\% = 29.925\%$

Cashew Nuts: $\frac{105}{100} \times 18.5\% = 19.425\%$

ACOD: $\frac{105}{100} \times 9.5\% = 9.975\%$

Table 4.3: Interest rates up by 5%

Investment	Interest Rate (%)	Risk Score (%)
Fixed Deposit	10.5	1.7
Treasury Bills	21.525	2.5
Cocoa Beans	21	1.5
Mortgage Securities	8.4	1.9
Construction Loans	29.925	2.9
Cashew Nuts	19.425	1.5
ACOD	9.975	2.2

As interest rates up by 5% only the technological coefficients of the decision variables changes while cost coefficients and the right hand side remains the same.

Thus the objective function becomes:

$$Z = 0.105x_1 + 0.21525x_2 + 0.21x_3 + 0.084x_4 + 0.29925x_5 + 0.19425x_6 + 0.09975x_7$$

Hence the problem is formulated as LP problem as follows:

***Maximise*:**

$$Z = 0.105x_1 + 0.21525x_2 + 0.21x_3 + 0.084x_4 + 0.29925x_5 + 0.19425x_6 + 09975x_7$$

***Subject to constraints* (4.1).**

Presentation of the solution

The technological coefficients of the revised function values and the original constraint function values were imputed into the computer and the Quantitative Manager software used to determine the output. The output is summarized as follows:

x_1 (Fixed Deposit) = Gh¢ 1,808,032.00

x_2 (Treasury Bills) = Gh¢ 3,674,300.00

x_3 (Cocoa Beans) = Gh¢ 1,974,300.00

x_4 (Mortgage Securities) = Gh¢ 813,768.00

x_5 (Construction Loans) = Gh¢ 3,155,800.00

x_6 (Cashew Nuts) = Gh¢ 1,674,800.00

x_7 (Apex Certificate of Deposit) = Gh¢ 1,900,000.00

The maximum return = **Gh¢ 2,983,096.55**

Thus each figure associated with each investment is the amount to commit to the investment in order to make a maximum return on the entire investment and the maximum return is **Gh¢ 2,983,096.55.** It is observed from the analysis above that as interest rates goes up by 5%, investment in Treasury Bills and in Construction Loans appreciates significantly. The expected return also experiences a tremendous increase by 17% (i.e, from **Gh¢ 2,553,096.55** to **Gh¢ 2,983,096.55**)

(*c*) The Firm's investments without Treasury Bills and Construction Loans

From the sensitivity analyses in (*a*) and (*b*) above it is observed that the rise and fall in interest rates has a dramatic effect on Treasury Bills and Construction Loans. We now solve the original LP problem (i.e. the Firm's Problem) without Treasury Bills and Construction Loans.

Table 4.4: Investments without Treasury Bills and Construction Loans

Investment	**Interest Rate**	**Risk Score (%)**
Fixed Deposit	10.0	1.7
Cocoa Beans	20.0	1.5
Mortgage Securities	8.0	1.9
Cashew Nuts	18.5	1.5
ACOD	9.5	2.2

The following present the notation and redefinition of the decision variables of the problem.

x_1 : Gh¢ to invest in Fixed Deposit

x_2 : Gh¢ to invest in Cocoa Beans

x_3 : Gh¢ to invest in Mortgage Securities

x_4 : Gh¢ to invest in Cashew Nuts

x_5 : Gh¢ to invest in ACOD

The resulting LP model is therefore as follows:

***Maximise*:**

$$Z = 0.1x_1 + 0.2x_2 + 0.08x_3 + 0.185x_4 + 0.095x_5$$

Subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 15,000,000$$

$$0.8x_1 - 0.2\,x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 \leq 0$$

$$0.2x_1 \;- 0.8\,x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 \geq 0$$

$$0.2x_1 \;+ 0.2\,x_2 - 0.8x_3 + 0.2x_4 + 0.2x_5 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 - 0.8\,x_4 + 0.2x_5 \geq 0$$

$$0.2x_1 + 0.2\,x_2 + 0.2x_3 + 0.2x_4 \;- 0.8x_5 \geq 0$$

$$0.3x_1 \;- 0.7x_2 + 0.3x_3 \;- 0.7x_4 + 0.3x_5 \leq 0$$

$$0.3x_1 + 0.5x_2 + 0.1x_3 + 0.5x_4 - 0.2x_5 \geq 0$$

$$0.75x_1 - 0.25x_2 \;- 0.25x_3 \;- 0.25x_4 - 0.25x_5 \geq 0$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots (4.2)$$

Presentation of the solution

The technological coefficients of the objective and constraint functions are imputed into the computer and the quantitative manager software used to determine the output. Below is the summary of the solution to the above LP problem:

x_1 (Fixed Deposit) = Gh¢ 3,000,000.00

x_2 (Cocoa Beans) = Gh¢ 3,000,000.00

x_3 (Mortgage Securities) = Gh¢ 2,700,000.00

x_4 (Cashew Nuts) = Gh¢ 3,000,000.00

x_5 (ACOD) = Gh¢ 3,300,000.00

The maximum return = **Gh¢ 1,895,092.65**

It is observed from the analysis above that if the firm decides not to invest in

Treasury Bills and Construction Loans, the expected return would be **Gh¢ 1,895,092.65,** a reduction by 25.8% (ie, from **Gh¢ 2,553,096.55** to **Gh¢ 1,895,092.65**).

(*d*) Redundancy of a constraint

The LP problem has seven decision variables and twelve constraints, thus producing twelve equations with seven unknowns. Under normal circumstances the number of decision variables must equal the number of constraints. It therefore follows that some of the constraints may not contribute to the solution of the LP problem; thus rendering them redundant. We therefore run the software without some of these constraints to determine the effect on the final solution.

(*e*) The Firm's investment s without the constraint 'AT LEAST 20% OF TOTAL INVESTMENT INTO DEPOSITS'.

The resulting LP model is therefore as follows:

Maximise:

$$Z = 0.1x_1 + 0.205x_2 + 0.2x_3 + 0.08x_4 + 0.285x_5 + 0.185x_6 + 0.095x_7$$

Subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 15,000,000$$

$$0.8x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 + 0.2x_6 + 0.2x_7 \leq 0$$

$$0.2x_1 - 0.8x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2x_2 - 0.8x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2x_2 + 0.2x_3 - 0.8x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2x_2 + 0.2x_3 + 0.2x_4 - 0.8x_5 + 0.2x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 - 0.8x_6 + 0.2x_7 \geq 0$$

$$0.2x_1 + 0.2x_2 + 0.2x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 - 0.8x_7 \geq 0$$

$$0.3x_1 + 0.3x_2 - 0.7x_3 + 0.3x_4 + 0.3x_5 - 0.7x_6 + 0.3x_7 \leq 0$$

$$0.45x_1 - 0.55x_2 + 0.45x_3 + 0.45x_4 - 0.55x_5 + 0.45x_6 + 0.45x_7 \leq 0$$

$$0.3x_1 - 0.5x_2 + 0.5x_3 + 0.1x_4 - 0.9x_5 + 0.5x_6 - 0.2x_7 \geq 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.3)$$

Presentation of the solution

The technological coefficients of the objective and constraint functions of the LP problem above are imputed into the computer and the Quantitative Manager Software used to determine the output. Below is the summary of the solution to the above LP problem above.

x_1 (Fixed Deposit) = Gh¢ 2,674,300.00

x_2 (Treasury Bills) = Gh¢ 2,555,000.00

x_3 (Cocoa Beans) = Gh¢ 1,974,800.00

x_4 (Mortgage Securities) = Gh¢ 813,480.00

x_5 (Construction Loans) = Gh¢ 2,407,320.00

x_6 (Cashew Nuts) = Gh¢ 2,675,000.00

x_7 (Apex Certificate of Deposit) = Gh¢ 1,900,500.00

The maximum return = Gh¢ 2,553,096.55

It is observed from the analysis above that the omitted constraint has no effect on the solution of the original LP problem of the Firm; thus rendering that constraint redundant. Hence the Firm could do without that constraint.

4.6 Summary of solutions

The solutions to the various scenarios considered (i.e, sensitivity analyses) are compared to the solution to the original LP problem resulting from the data supplied by the firm under consideration. Table 4.5 below shows the comparison of rate on investments and return on investments before and after sensitivity analyses. From the table, as interest rate on each investment down by 5%, the return on the portfolio decreases from **Gh¢ 2,553,096.55** to **Gh¢ 2,170,132.068** (i.e, a decrease of about 15%). However, return on the portfolio saw an increased from **Gh¢ 2,553,096.55** to **Gh¢ 2,983,096.55** (i.e, an increase of about 17%). **Gh¢ 1,895,092.65** is the expected return on the portfolio without Treasury Bills and Construction Loans being components of the portfolio. Next, from table 4.6, as interest rates of the original data increases by 5%, return on investments also grows by almost 17% (ie from Gh¢ 2,553,096.55 to Gh¢ 2,983,096.55) with quantum of money on Treasury Bills and Construction Loans being the most affected in terms of increment. However the quantum of money on Fixed Deposit, Cocoa Beans, Cashew Nuts and ACOD experienced a decrease while Mortgage Securities record a slight increase. Again from table 4.6, when Treasury Bills and Construction Loans are exempted from the portfolio, the quantum of money on each investment increases rapidly while return on portfolio reduces to

Gh¢ 1,895,092.65.

Table 4.5 Comparison of rate on investment and return on investments before and after sensitivity analyses

Investment	Rate on Investment (%)	Rate on Investment down by 5% (%)	Rate on Investment up by 5% (%)	Rate on Investment (without Treasury Bills and Construction Loans) (%)
Fixed Deposit	10.0	9.5	10.5	10.0
Treasury Bills	20.5	19.475	21.0	–
Cocoa Beans	20.0	19	20.5	20.0
Mortgage Securities	8.0	7.6	8.5	8.0
Construction Loans	28.5	27.075	29.0	–
Cashew Nuts	18.5	17.575	19.0	18.5
Apex Certificate of Deposit	9.5	9.025	10.0	9.5
Return on investments (Gh¢)	**2,553,096.55**	**2,170,132.068**	**2,983,096.55**	**1,895,092.65**

Table 4.6 Comparison of quantum of money on investment and return on investments before and after sensitivity analyses

Investment	Quantum of Money on Investment (Original Interest Rate) (Gh¢)	Quantum of Money on Investment (Interest Rate down by 5%) (Gh¢)	Quantum Of Money on Investment (Interest Rate up by 5%) (Gh¢)	Quantum of Money on Original Investment (without Treasury Bills and Construction Loans) (Gh¢)
Fixed Deposit	2,674,300	2,399,800	1,808,032	3,000,000
Treasury Bills	2,554,800	1,874,300	3,674,300	–
Cocoa Beans	1,974,800	2,129,800	1,974,300	3,000,000
Mortgage Securities	813,480	2,207,320	813,768	2,700,000
Construction Loans	2,407,320	1,013,480	3,155,800	–
Cashew Nuts	2,674,800	2,674,800	1,674,800	3,000,000
Apex Certificate of Deposit	1,9005,00	2,700,500	1,900,000	3,300,000
Return on investments (Gh¢)	**2,553,096.55**	**2,170,132.068**	**2,983,096.55**	**1,895,092.65**

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

The research considered an optimal portfolio mixed for a firm with the aim of maximising return while operating within the firm's operational policies and to determine how much money should be invested in each investment area identified to maximise returns using linear programming and the Quantitative Manager. This chapter looks at the conclusions and recommendations.

5.2 Conclusions

From table 4.4, as interest rates on the original data reduces by 5%, return on investments also goes down by almost 15% (ie from Gh¢ 2,553,096.55 to Gh¢ 2,170,132.068). Also the quantum of money on Treasury Bills and Construction Loans saw a significant reduction from Gh¢ 2,554,800.00 to Gh¢ 1,874,300.00 and from Gh¢ 2,407,320.00 to Gh¢ 1,013,480.00 respectively for maximum return. However investment in Cocoa Beans, Mortgage Securities, Cashew Nuts and Apex Certificate of Deposit (ACOD) appreciated from Gh¢ 1,974,800.00 to Gh¢ 2,129,800.00 from Gh¢ 813,480.00 to Gh¢ 2,207,320.00 from Gh¢ 2,674,800.00 to Gh¢ 2,674,800.00 and from Gh¢ 1,900,500.00 to Gh¢ 2,7005,00.00 respectively. But Fixed Deposit saw a slight decrease from Gh¢ 2, 674, 300.00 to Gh¢ 2,399,800.00.

Next, from table 6, as interest rates of the original data increases by 5%, return on investments also grows by almost 17% (ie from Gh¢ 2,553,096.55 to Gh¢ 2,983,096.55) with quantum of money on Treasury Bills and Construction Loans being the most affected in terms of increment; that is increasing from Gh¢ 2,554,800.00 to Gh¢ 3,674,300.00 and from Gh¢ 2,407,320.00 to Gh¢ 3,155,800.00 respectively. However the quantum of money on Fixed Deposit, Cocoa Beans, Cashew Nuts and Apex Certificate of Deposit experienced a decrease from Gh¢ 2,674,300.00 to Gh¢ 1,808,032.00 Gh¢ 1,974,800.00 to Gh¢ 1,974,300.00 Gh¢ 2,674,800.00 to Gh¢ 1,674,800.00 and Gh¢ 1,900,500.00 to Gh¢ 1,900,000.00 respectively while Mortgage Securities record a slight increase from Gh¢ 813,480.00 to Gh¢ 813,768.00. Again from table 4.5, when Treasury Bills and Construction Loans are exempted from the portfolio, the quantum of money on each investment increases rapidly while return on investments reduces to Gh¢ 1,895,092.65.

Note

It is observed from the sensitivity analyses that investment in Treasury Bills and Construction Loans are major determinants for maximisation or otherwise of return on the investment portfolio.

5.3 Recommendations

The measurement and management of risk is a central issue in finance and a huge effort is made in order to analyse it and to understand all the related problems.

Optimal portfolio selection from the developing economies point of view should hinge on three things; Asset selection, asset allocation and investment implementation. The selection can add extra returns to selected portfolios. Selection requires significant research and it is costly. Therefore careful analysis of investment selected for a particular portfolio may add to active portfolio management. It is therefore recommended that asset classes are scrutinised before adding to the portfolio.

The allocation of investment to different asset classes is also necessary in optimal portfolio selection. Asset allocation is based on the idea that in different periods a different asset is the best-performing one. It is difficult to predict which asset will perform best in a given year. Therefore, although it is psychologically appealing to try to predict the best performing asset, it makes sense to carefully do the allocation based on the regulatory framework. This research finding confirm the widely held belief that market returns and asset allocation policy in excess of market return are collectively the dominant determinant in total return variations.

Accordingly, it is recommended that asset allocation should be the mix of asset classes that promises the highest long-term expected investment return. The use of Linear Programming is strongly recommended in determining the possible optimal investment in a portfolio mix. The use of the Quantitative Manager (QM) is crucial in analys ing the firm's data due to its flexibility in operating precision.

That notwithstanding, interpreting the printout of the Quantitative Manager after analysis and making recommendations may not be the very best of decision. Sensitivity analysis should be done on the available data to determine the stability of the model that would be obtained from the firm's operational policies and available funds and to also determine the major determinants of the decision variables of the firm for optimal return on investments.

The research also stresses the importance of examining investment behaviour from their overall risky asset composition. It also proposes further research that will reveal the interaction relationship of portfolio selections between risky financial assets and non-risky financial assets in the analysis. It is important to analyze whether certain types of investments are risky and may affect the entire portfolio. Future research should use a more advanced technique that will allow for endogeneity of the risky investment variables.

Future research should make more efforts on investigating how the components of various assets with the associated risk may also affect portfolio construction. A better understanding of portfolio choice through optimal selection may also be important to policymakers and practitioners for a number of reasons. It will aid in financial planning by the firm in relation to risky asset composition.

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APPENDICES

Table A1: Solution to the LP problem

	X_1	X_2	X_3	X_4	X_5	X_6	X_7		RHS
Max.	0.1	0.205	0.2	0.08	0.285	0.185	0.095		
Const 1	1	1	1	1	1	1	1	≤	15,000,000
Const 2	0.8	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	≤	0
Const 3	0.2	– 0.8	0.2	0.2	0.2	0.2	0.2	≥	0
Const 4	0.2	0.2	– 0.8	0.2	0.2	0.2	0.2	≥	0
Const 5	0.2	0.2	0.2	– 0.8	0.2	0.2	0.2	≥	0
Const 6	0.2	0.2	0.2	0.2	– 0.8	0.2	0.2	≥	0
Const 7	0.2	0.2	0.2	0.2	0.2	– 0.8	0.2	≥	0
Const 8	0.2	0.2	0.2	0.2	0.2	0.2	– 0.8	≥	0
Const 9	0.75	– 0.25	– 0.25	– 0.25	– 0.25	– 0.25	0.75	≥	0
Cons 10	0.3	0.3	– 0.7	0.3	0.3	– 0.7	0.3	≤	0
Cons 11	0.45	– 0.55	0.45	0.45	– 0.55	0.45	0.45	≤	0
Cons 12	0.3	– 0.5	0.5	0.1	– 0.9	0.5	– 0.2	≥	0
SOLTION	2,674,300	2,554,800	1,974,800	813,480	2,407,320	2,674,800	1,900,500		2,553,096.55

Table A 2: Solution to the LP problem when interest rates are down by 5%

	X_1	X_2	X_3	X_4	X_5	X_6	X_7		RHS
Max.	0.095	0.19475	0.19	0.076	0.27075	0.17575	0.09025		
Const 1	1	1	1	1	1	1	1	≤	15,000,000
Const 2	0.8	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	≤	0
Const 3	0.2	– 0.8	0.2	0.2	0.2	0.2	0.2	≥	0
Const 4	0.2	0.2	– 0.8	0.2	0.2	0.2	0.2	≥	0
Const 5	0.2	0.2	0.2	– 0.8	0.2	0.2	0.2	≥	0
Const 6	0.2	0.2	0.2	0.2	– 0.8	0.2	0.2	≥	0
Const 7	0.2	0.2	0.2	0.2	0.2	– 0.8	0.2	≥	0
Const 8	0.2	0.2	0.2	0.2	0.2	0.2	– 0.8	≥	0
Const 9	0.75	– 0.25	– 0.25	– 0.25	– 0.25	– 0.25	0.75	≥	0
Cons 10	0.3	0.3	– 0.7	0.3	0.3	– 0.7	0.3	≤	0
Cons 11	0.45	– 0.55	0.45	0.45	– 0.55	0.45	0.45	≤	0
Cons 12	0.3	– 0.5	0.5	0.1	– 0.9	0.5	– 0.2	≥	0
SOLTION	2,399,800	1,874,300	2,129,800	2,207,320	1,013,480	2,674,800	2,700,500		2,553,096.55

Table A3: Solution to the LP problem when interest rates are up by 5%

	X_1	X_2	X_3	X_4	X_5	X_6	X_7		RHS
Max.	0.105	0.21525	0.21	0.084	0.29925	0.19425	0.09975		
Const 1	1	1	1	1	1	1	1	≤	15,000,000
Const 2	0.8	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	≤	0
Const 3	0.2	– 0.8	0.2	0.2	0.2	0.2	0.2	≥	0
Const 4	0.2	0.2	– 0.8	0.2	0.2	0.2	0.2	≥	0
Const 5	0.2	0.2	0.2	– 0.8	0.2	0.2	0.2	≥	0
Const 6	0.2	0.2	0.2	0.2	– 0.8	0.2	0.2	≥	0
Const 7	0.2	0.2	0.2	0.2	0.2	– 0.8	0.2	≥	0
Const 8	0.2	0.2	0.2	0.2	0.2	0.2	– 0.8	≥	0
Const 9	0.75	– 0.25	– 0.25	– 0.25	– 0.25	– 0.25	0.75	≥	0
Cons 10	0.3	0.3	– 0.7	0.3	0.3	– 0.7	0.3	≤	0
Cons 11	0.45	– 0.55	0.45	0.45	– 0.55	0.45	0.45	≤	0
Cons 12	0.3	– 0.5	0.5	0.1	– 0.9	0.5	– 0.2	≥	0
SOLTION	1,808,032	3,674,300	1,974,300	813,768	3,155,800	1,674,800	1,900,000		2,983,096.55

Table A4: Solution to the LP problem without Treasury Bills and Construction Loans

	X_1	X_2	X_3	X_4	X_5		RHS
Max.	**0.1**	**0.2**	**0.08**	**0.185**	**0.095**		
Const 1	**1**	**1**	**1**	**1**	**1**	≤	**15,000,000**
Const 2	**0.8**	**– 0.2**	**– 0.2**	**– 0.2**	**– 0.2**	≤	**0**
Const 3	**0.2**	**– 0.8**	**0.2**	**0.2**	**0.2**	≥	**0**
Const 4	**0.2**	**0.2**	**– 0.8**	**0.2**	**0.2**	≥	**0**
Const 5	**0.2**	**0.2**	**0.2**	**– 0.8**	**0.2**	≥	**0**
Const 6	**0.2**	**0.2**	**0.2**	**0.2**	**– 0.8**	≥	**0**
Const 7	**0.3**	**– 0.7**	**0.3**	**– 0.7**	**0.3**	≤	**0**
Const 8	**0.3**	**0.5**	**0.1**	**0.5**	**-0.2**	≥	**0**
Const 9	**0.75**	**– 0.25**	**– 0.25**	**– 0.25**	**– 0.25**	≥	**0**
SOLTION	**3,000,000**	**3,000,000**	**2,700,000**	**3,000,000**	**3,300,000**		**1,895,092.65**

Table A5: Solution to the LP problem without the constraint 'AT LEAST 20 % OF TOTAL INVESTMENT INTO DEPOSITS'.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7		RHS
Max.	0.1	0.205	0.2	0.08	0.285	0.185	0.095		
Const 1	1	1	1	1	1	1	1	≤	15,000,000
Const 2	0.8	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	– 0.2	≤	0
Const 3	0.2	– 0.8	0.2	0.2	0.2	0.2	0.2	≥	0
Const 4	0.2	0.2	– 0.8	0.2	0.2	0.2	0.2	≥	0
Const 5	0.2	0.2	0.2	– 0.8	0.2	0.2	0.2	≥	0
Const 6	0.2	0.2	0.2	0.2	– 0.8	0.2	0.2	≥	0
Const 7	0.2	0.2	0.2	0.2	0.2	– 0.8	0.2	≥	0
Const 8	0.2	0.2	0.2	0.2	0.2	0.2	– 0.8	≥	0
Const 9	0.3	0.3	– 0.7	0.3	0.3	– 0.7	0.3	≥	0
Cons 10	0.45	– 0.55	0.45	0.45	– 0.55	0.45	0.45	≤	0
Cons 11	0.3	– 0.5	0.5	0.1	– 0.9	0.5	– 0.2	≥	0
SOLTION	2,674,100	2,555,000	1,974,800	813,480	2,407,120	2,675,000	1,900,500		2,553,096.55