SOME INEQUALITIES FOR THE RATIOS OF GENERALIZED DIGAMMA FUNCTIONS

KWARA NANTOMAH

Department of Mathematics, University for Development Studies, Navrongo Campus, P.O. Box 24, Navrongo, UE/R, Ghana

Copyright © 2014 Kwara Nantomah. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, some inequalities for the ratios of generalized digamma functions are presented. The approach makes use of the series representations of the \((q,k)\)-digamma and \((p,q)\)-digamma functions.

Keywords: digamma function; \((q,k)\)-digamma function; \((p,q)\)-digamma function; inequality.

2010 AMS Subject Classification: 33B15, 26A48.

1. Introduction and preliminaries

The classical Euler’s Gamma function \(\Gamma(t)\) and the digamma function \(\psi(t)\) are commonly defined as

\[
\Gamma(t) = \int_0^\infty e^{-x}x^{t-1} \, dx, \quad \psi(t) = \frac{d}{dt} \ln \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.
\]

In 2005, Díaz and Teruel [1] defined the \((q,k)\)-Gamma function, \(\Gamma_{q,k}(t)\) as

\[
\Gamma_{q,k}(t) = \frac{(1-q^k)^{\frac{t}{k}}}{(1-q)^{\frac{t}{k}}}, \quad t > 0, k > 0, q \in (0,1)
\]

Received May 21, 2014
with the \((q,k)\)-digamma function, \(\psi_{q,k}(t)\) is defined as

\[
\psi_{q,k}(t) = \frac{d}{dt} \ln \Gamma_{q,k}(t) = \frac{\Gamma'_{q,k}(t)}{\Gamma_{q,k}(t)}, \quad t > 0, \, k > 0, \, q \in (0,1).
\]

Also in 2012, Krasniqi and Merovci [2] gave the \((p,q)\)-Gamma function, \(\Gamma_{p,q}(t)\) as

\[
\Gamma_{p,q}(t) = \frac{[p]_q! [p]_q}{[t]_q [t+1]_q \ldots [t+p]_q}, \quad t > 0, \, p \in \mathbb{N}, \, q \in (0,1),
\]

where \( [p]_q = \frac{1-q^p}{1-q} \).

Similarly, the \((p,q)\)-digamma function, \(\psi_{p,q}(t)\) is defined as

\[
\psi_{p,q}(t) = \frac{d}{dt} \ln \Gamma_{p,q}(t) = \frac{\Gamma'_{p,q}(t)}{\Gamma_{p,q}(t)}, \quad t > 0, \, p \in \mathbb{N}, \, q \in (0,1).
\]

The functions \(\psi_{q,k}(t)\) and \(\psi_{p,q}(t)\) as defined above exhibit the following series representations.

\[
\psi_{q,k}(t) = \frac{-\ln(1-q)}{k} + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nk} t}{1-q^{nk}}, \quad t > 0,
\]

\[
\psi_{p,q}(t) = \ln [p]_q + (\ln q) \sum_{n=1}^{p} \frac{q^{nt}}{1-q^n}, \quad t > 0.
\]

By taking derivatives of these functions, it can easily be established that

\[
\psi'_{q,k}(t) = (\ln q)^2 \sum_{n=1}^{\infty} \frac{nkq^{nk} t}{1-q^{nk}}, \quad t > 0,
\]

\[
\psi'_{p,q}(t) = (\ln q)^2 \sum_{n=1}^{p} \frac{nq^{nt}}{1-q^n}, \quad t > 0.
\]

In [3], Nantomah presented the following results for the digamma function.

\[
\frac{[\psi(a)]^\alpha}{[\psi(c)]^\beta} \leq \frac{[\psi(a+bt)]^\alpha}{[\psi(c+dt)]^\beta} \leq \frac{[\psi(a+b)]^\alpha}{[\psi(c+d)]^\beta},
\]

where \(a, b, c, d, \alpha, \beta\) are positive real numbers such that \(\beta d \leq \alpha b, a+bt \leq c+dt, \psi(a+bt) > 0\) and \(\psi(c+dt) > 0\). The \(k\)-analogue of these inequalities can be found in [4].

The purpose of this paper is to extend inequalities (5) to the \((q,k)\) and \((p,q)\)-digamma functions.

### 2. Results and discussion

We now present the results of this paper.
Lemma 2.1. Let $0 < s \leq t$, then the following statement is valid.

$$\psi_{q,k}(s) \leq \psi_{q,k}(t).$$

Proof. From (1), we have

$$\psi_{q,k}(s) - \psi_{q,k}(t) = (\ln q) \sum_{n=1}^{\infty} \left[ \frac{q^{nks} - q^{nkt}}{1 - q^{nk}} \right] \leq 0.$$ 

Lemma 2.2. Let $0 < s \leq t$, then the following statement is valid.

$$\psi'_{q,k}(s) \geq \psi'_{q,k}(t).$$

Proof. From (3) we have,

$$\psi'_{q,k}(s) - \psi'_{q,k}(t) = (\ln q)^2 \sum_{n=1}^{\infty} \left[ \frac{nk(q^{nks} - q^{nkt})}{1 - q^{nk}} \right] \geq 0.$$ 

Lemma 2.3. Let $a, b, c, d, \alpha, \beta$ be positive real numbers such that $a + bt \leq c + dt$, $\beta d \leq \alpha b$, $\psi_{q,k}(a + bt) > 0$ and $\psi_{q,k}(c + dt) > 0$. Then

$$\alpha b \psi_{q,k}(c + dt) \psi'_{q,k}(a + bt) - \beta d \psi_{q,k}(a + bt) \psi'_{q,k}(c + dt) \geq 0.$$ 

Proof. Since $0 < a + bt \leq c + dt$, then by Lemmas 2.1 and 2.2 we have

$$0 < \psi_{q,k}(a + bt) \leq \psi_{q,k}(c + dt)$$

and

$$\psi'_{q,k}(a + bt) \geq \psi'_{q,k}(c + dt) > 0.$$ 

This implies

$$\psi_{q,k}(c + dt) \psi'_{q,k}(a + bt) \geq \psi_{q,k}(c + dt) \psi'_{q,k}(c + dt) \geq \psi_{q,k}(a + bt) \psi'_{q,k}(c + dt).$$

Further, $\alpha b \geq \beta d$ implies

$$\alpha b \psi_{q,k}(c + dt) \psi'_{q,k}(a + bt) \geq \alpha b \psi_{q,k}(a + bt) \psi'_{q,k}(c + dt) \geq \beta d \psi_{q,k}(a + bt) \psi'_{q,k}(c + dt).$$
Hence, we have
\[ \alpha b \psi_{q,k}^\prime (c + dt) \psi_{q,k}^\prime (a + bt) - \beta d \psi_{q,k} (a + bt) \psi_{q,k}^\prime (c + dt) \geq 0. \]

**Theorem 2.4.** Define a function \( G \) by
\[
(6) \quad G(t) = \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta}, \quad t \in [0, \infty),
\]
where \( a, b, c, d, \alpha, \beta \) are positive real numbers such that \( a + bt \leq c + dt \), \( \beta d \leq \alpha b \), \( \psi_{q,k}(a + bt) \geq 0 \) and \( \psi_{q,k}(c + dt) \geq 0 \). Then \( G \) is nondecreasing on \( t \in [0, \infty) \) and the inequalities
\[
(7) \quad \frac{[\psi_{q,k}(a)]^\alpha}{[\psi_{q,k}(c)]^\beta} \leq \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta} \leq \frac{[\psi_{q,k}(a+b)]^\alpha}{[\psi_{q,k}(c+d)]^\beta}
\]
are valid for every \( t \in [0, 1] \).

**Proof.** Let \( g(t) = \ln G(t) \) for every \( t \in [0, \infty) \). Then,
\[
g = \ln \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta} = \alpha \ln \psi_{q,k}(a+bt) - \beta \ln \psi_{q,k}(c+dt)
\]
and
\[
g'(t) = \alpha b \frac{\psi_{q,k}^\prime (a+bt)}{\psi_{q,k}(a+bt)} - \beta d \frac{\psi_{q,k}^\prime (c+dt)}{\psi_{q,k}(c+dt)}
\]
\[
= \frac{\alpha b \psi_{q,k}^\prime (a+bt) \psi_{q,k} (c+dt) - \beta d \psi_{q,k} (c+dt) \psi_{q,k}^\prime (a+bt)}{\psi_{q,k} (a+bt) \psi_{q,k} (c+dt)} \geq 0
\]
as a result of Lemma 2.3. That implies \( g \) as well as \( G \) are nondecreasing on \( t \in [0, \infty) \) and for every \( t \in [0, 1] \) we have
\[
G(0) \leq G(t) \leq G(1)
\]
concluding the proof.

**Corollary 2.5.** If \( t \in (1, \infty) \), then the following inequality is valid.
\[
(8) \quad \frac{[\psi_{q,k}(a+bt)]^\alpha}{[\psi_{q,k}(c+dt)]^\beta} \geq \frac{[\psi_{q,k}(a+b)]^\alpha}{[\psi_{q,k}(c+d)]^\beta}
\]

**Proof.** For each \( t \in (1, \infty) \), we have \( G(t) \geq G(1) \) yielding the result.

**Lemma 2.6.** Let \( 0 < s \leq t \), then the following statement is valid.
\[
\psi_{p,q}(s) \leq \psi_{p,q}(t).
\]
Proof. From (2) we have
\[ \psi_{p,q}(s) - \psi_{p,q}(t) = (\ln q)^2 \sum_{n=1}^{p} \frac{q^{ns} - q^{nt}}{1 - q^n} \leq 0. \]

Lemma 2.7. Let \(0 < s \leq t\), then the following statement is valid.
\[ \psi'_{p,q}(s) \geq \psi'_{p,q}(t). \]

Proof. From (4) we have
\[ \psi'_{p,q}(s) - \psi'_{p,q}(t) = (\ln q)^2 \sum_{n=1}^{p} \frac{n(q^{ns} - q^{nt})}{1 - q^n} \geq 0. \]

Lemma 2.8. Let \(a, b, c, d, \alpha, \beta\) be positive real numbers such that \(a + bt \leq c + dt\), \(\beta d \leq \alpha b\), \(\psi_{p,q}(a + bt) > 0\) and \(\psi_{p,q}(c + dt) > 0\). Then
\[ \alpha b \psi_{p,q}(c + dt) \psi'_{p,q}(a + bt) - \beta d \psi_{p,q}(a + bt) \psi'_{p,q}(c + dt) \geq 0. \]

Proof. Follows the same argument as in the proof of Lemma 2.3.

Theorem 2.9. Define a function \(H\) by
\[ H(t) = \left[ \frac{\psi_{p,q}(a + bt)}{\psi_{p,q}(c + dt)} \right]^\alpha, \quad t \in [0, \infty), \]
where \(a, b, c, d, \alpha, \beta\) are positive real numbers such that \(a + bt \leq c + dt\), \(\beta d \leq \alpha b\), \(\psi_{p,q}(a + bt) > 0\) and \(\psi_{p,q}(c + dt) > 0\). Then \(H\) is nondecreasing on \(t \in [0, \infty)\) and the inequalities
\[ \left[ \frac{\psi_{p,q}(a)}{\psi_{p,q}(c)} \right]^\alpha \leq \left[ \frac{\psi_{p,q}(a + bt)}{\psi_{p,q}(c + dt)} \right]^\alpha \leq \left[ \frac{\psi_{p,q}(a + b)}{\psi_{p,q}(c + d)} \right]^\beta \]
are valid for every \(t \in [0, 1]\).

Proof. Follows the same procedure as in Theorem 2.4. Using Lemma 2.3, we conclude that \(H\) is nondecreasing on \(t \in [0, \infty)\) and for every \(t \in [0, 1]\) we have, \(H(0) \leq H(t) \leq H(1)\) ending the proof.

Corollary 2.10. If \(t \in (1, \infty)\), then the following inequality is valid.
\[ \frac{\psi_{p,q}(a + bt)}{\psi_{p,q}(c + dt)}^\beta \geq \frac{\psi_{p,q}(a + b)}{\psi_{p,q}(c + d)}^\beta \]

Proof. For each \(t \in (1, \infty)\), we have \(H(t) \geq H(1)\) yielding the result.
3. Concluding remarks

This section is dedicated to some remarks concerning our results.

**Remark 3.1.** If in (7) we allow $k \to 1$, then we obtain the $q$-analogue of (5).

**Remark 3.2.** If in (7) we allow $q \to 1^-$, then we obtain the $k$-analogue of (5) as presented in Theorem 3.7 of the paper [4].

**Remark 3.3.** If in (7) we allow $q \to 1^-$ as $k \to 1$, then we obtain (5).

**Remark 3.4.** If in (10) we allow $q \to 1^-$, then we obtain the $p$-analogue of (5).

**Remark 3.5.** If in (10) we allow $p \to \infty$, then we obtain the $q$-analogue of (5).

**Remark 3.6.** If in (10) we allow $p \to \infty$ as $q \to 1^-$, then we obtain (5).

**Conflict of Interests.**

The author declares that there is no conflict of interests.

**REFERENCES**


