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On a (p,k)-analogue of the Gamma function and some associated Inequalities

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Abstract. In this paper, we introduce a new two-parameter deformation of the classical Gamma function, which we call a (p,k)-analogue of the Gamma function. We also provide some identities generalizing those satisfied by the classical Gamma function. Furthermore, we establish some inequalities involving this new function.

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1. Introduction

The classical Euler's Gamma function, $\Gamma(x)$ is usually defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt=\lim_{n\to\infty}\frac{n!n^x}{x(x+1)(x+2)\dots(x+n)}$$

It is well-known that $\Gamma(x)$ satisfies the following basic relations.

$$\Gamma(n+1)=n!,\quad n\in\mathbb{Z}^+\cup\{0\},$$
$$\Gamma(x+1)=x\Gamma(x),\quad x\in\mathbb{R}^+.$$

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Closely associated with the Gamma function is the Digamma or Psi function $\psi(x)$, which is defined for $x > 0$ as the logarithmic derivative of the Gamma function. That is,

$$\begin{aligned}\psi(x) &= \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},\\ &= -\gamma + (x-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+x)},\\ &= -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}\end{aligned}$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.577215664...$ is the Euler-Mascheroni's constant. The Polygamma functions, $\psi^{(m)}(x)$ are defined for $x > 0$ and $m \in \mathbb{N}$ as

$$\begin{aligned}\psi^{(m)}(x) &= \frac{d^m}{dx^m}\psi(x) = \frac{d^{m+1}}{dx^{m+1}}\ln\Gamma(x)\\ &= (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{m+1}}\end{aligned}$$

where $\psi^{(0)}(x) \equiv \psi(x)$.

The p-analogue (also known as p-extension or p-deformation) of the Gamma function is defined for $p \in \mathbb{N}$ and $x > 0$ as

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\dots(x+p)}$$

where $\lim_{p\to\infty}\Gamma_p(x) = \Gamma(x)$. See [1, p. 270]. It satisfies the identities:

$$\begin{aligned}\Gamma_p(x+1) &= \frac{px}{x+p+1}\Gamma_p(x),\\ \Gamma_p(1) &= \frac{p}{p+1}.\end{aligned}$$

The p-analogues of the Digamma and Polygamma functions are defined for $x > 0$ as

$$\begin{aligned}\psi_p(x) &= \frac{d}{dx}\ln\Gamma_p(x) = \ln p - \sum_{n=0}^{p}\frac{1}{n+x},\\ \psi_p^{(m)}(x) &= \frac{d^m}{dx^m}\psi_p(x) = (-1)^{m-1}m!\sum_{n=0}^{p}\frac{1}{(n+x)^{m+1}}\end{aligned}$$

where $\psi_p^{(0)}(x) \equiv \psi_p(x)$.

In 2007, Díaz and Pariguan [2] also defined the k-analogue of the Gamma function for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}^-$ as

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

where $\lim_{k\to 1} \Gamma_k(x) = \Gamma(x)$ and $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$ is the Pochhammer k-symbol. The k-analogue also satisfies the identities:

$$\begin{aligned}\Gamma_k(x+k) &= x\Gamma_k(x), \quad x \in R^+\\ \Gamma_k(k) &= 1\end{aligned}$$

Similarly, the k-analogues of the Digamma and Polygamma functions are defined for $x > 0$ as

$$\begin{aligned}\psi_k(x) &= \frac{d}{dx}\ln\Gamma_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(nk+x)},\\ \psi_k^{(m)}(x) &= \frac{d^m}{dx^m}\psi_k(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(nk+x)^{m+1}}\end{aligned}$$

where $\psi_k^{(0)}(x) \equiv \psi_k(x)$.

The purpose of this paper is to introduce a new two-parameter deformation of the classical Gamma function, called a (p,k)-analogue of the Gamma function. In addition, we provide some identities and inequalities involving this function. We present our results in the following section.

2. Results and Discussion

Definition 2.1. *Let $p \in \mathbb{N}$ and $k > 0$. Then the (p,k)-analogue (also called the (p,k)-deformation or (p,k)-generalization) of the Gamma function is defined as*

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1}\left(1 - \frac{t^k}{pk}\right)^p dt \tag{1}$$

$$= \frac{(p+1)! k^{p+1} (pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)} \tag{2}$$

for $x \in \mathbb{R}^+$. It satisfies the identities:

$$\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \tag{3}$$

$$\Gamma_{p,k}(ak) = \frac{p+1}{p} k^{a-1}\Gamma_p(a), \quad a \in \mathbb{R}^+ \tag{4}$$

$$\Gamma_{p,k}(k) = 1. \tag{5}$$

Also, observe that $\Gamma_{p,k}(x)$ satisfies the following commutative diagram.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p\to\infty} & \Gamma_k(x) \\ {\scriptstyle k\to 1}\downarrow & & \downarrow{\scriptstyle k\to 1} \\ \Gamma_p(x) & \xrightarrow[p\to\infty]{} & \Gamma(x) \end{array}$$

The (p,k)-analogue of the Digamma function is defined as the logarithmic derivative of $\Gamma_{p,k}(x)$. That is

$$\psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k}(x) = \frac{\Gamma'_{p,k}(x)}{\Gamma_{p,k}(x)}.$$

The function $\psi_{p,k}(x)$ satisfies the following series and integral representations.

$$\psi_{p,k}(x) = \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{(nk+x)} \tag{6}$$

$$= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt \tag{7}$$

The (p,k)-analogue of the Polygamma functions are defined as

$$\psi^{(m)}_{p,k}(x) = \frac{d^m}{dx^m}\psi_{p,k}(x) = (-1)^{m+1}m!\sum_{n=0}^{p}\frac{1}{(nk+x)^{m+1}} \tag{8}$$

$$= (-1)^{m+1}\int_0^\infty \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right)t^m e^{-xt}\,dt \tag{9}$$

for $m \in \mathbb{N}$, where $\psi^{(0)}_{p,k}(x) \equiv \psi_{p,k}(x)$. It follows easily from (8) that,

$$\psi^{(m)}_{p,k}(x) = \begin{cases} > 0 & \text{if } m \text{ is odd} \\ < 0 & \text{if } m \text{ is even,} \end{cases} \tag{10}$$

which means that the function $\psi'_{p,k}(x)$ is a completely monotonic function of x, for $x \in \mathbb{R}^+$.

Remark 2.1. *From the identity (3), we obtain the relations*

$$\psi_{p,k}(x+k) - \psi_{p,k}(x) = \frac{1}{x} - \frac{1}{x+pk+k}, \tag{11}$$

$$\psi^{(m)}_{p,k}(x+k) - \psi^{(m)}_{p,k}(x) = \frac{(-1)^m m!}{x^{m+1}} - \frac{(-1)^m m!}{(x+pk+k)^{m+1}}, \quad m \in \mathbb{N}. \tag{12}$$

Also from (6), we obtain the relation

$$\psi_{p,k}(k) = \frac{1}{k}\left[\ln(pk) - H(p+1)\right]$$

where $H(n)$ is the nth harmonic number.

The (p,k)-analogue of the classical Beta function is defined as

$$B_{p,k}(x,y)=\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)},\quad x>0, y>0.$$

Lemma 2.1. *The function $\psi_{p,k}(x)$ satisfies the following limit properties.*
(i) $\psi_{p,k}(x)\to\psi_k(x)$ as $p\to\infty$,
(ii) $\psi_{p,k}(x)\to\psi_p(x)$ as $k\to 1$,
(iii) $\psi_{p,k}(x)\to\psi(x)$ as $p\to\infty$ and $k\to 1$.

Proof. (i) By (6), we have

$$\begin{aligned}\lim_{p\to\infty}\psi_{p,k}(x)&=\lim_{p\to\infty}\left[\frac{1}{k}\ln(pk)-\frac{1}{x}-\sum_{n=1}^{p}\frac{1}{(nk+x)}-\sum_{n=1}^{p}\frac{1}{nk}+\sum_{n=1}^{p}\frac{1}{nk}\right]\\&=\lim_{p\to\infty}\left[\frac{1}{k}\ln(pk)-\sum_{n=1}^{p}\frac{1}{nk}-\frac{1}{x}+\sum_{n=1}^{p}\frac{1}{nk}-\sum_{n=1}^{p}\frac{1}{(nk+x)}\right]\\&=\lim_{p\to\infty}\left[\frac{1}{k}\ln k+\frac{1}{k}\ln p-\frac{1}{k}\sum_{n=1}^{p}\frac{1}{n}-\frac{1}{x}+\sum_{n=1}^{p}\frac{x}{nk(nk+x)}\right]\\&=\frac{1}{k}\ln k+\frac{1}{k}\lim_{p\to\infty}\left[\ln p-\sum_{n=1}^{p}\frac{1}{n}\right]-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{nk(nk+x)}\\&=\frac{1}{k}\ln k-\frac{\gamma}{k}-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{nk(nk+x)}\\&=\psi_k(x).\end{aligned}$$

(ii) Also by (6), we have

$$\lim_{k\to1}\psi_{p,k}(x)=\lim_{k\to1}\left[\frac{1}{k}\ln(pk)-\sum_{n=0}^{p}\frac{1}{(nk+x)}\right]=\ln p-\sum_{n=0}^{p}\frac{1}{(n+x)}=\psi_p(x).$$

(iii) This follows easily from (i) or (ii). That is,

$$\lim_{k\to1}\left(\lim_{p\to\infty}\psi_{p,k}(x)\right)=\lim_{k\to1}\psi_k(x)=\psi(x),\quad\text{or}$$
$$\lim_{p\to\infty}\left(\lim_{k\to1}\psi_{p,k}(x)\right)=\lim_{p\to\infty}\psi_p(x)=\psi(x).$$

□

Lemma 2.2. *Let $\gamma_{p,k}=-\psi_{p,k}(1)$ be the (p,k)-analogue of the Euler-Mascheroni's constant. Then $\gamma_{p,k}\to\gamma$ as $p\to\infty$ and $k\to 1$.*

Proof. Proceed as follows

$$\lim_{k\to1}\psi_{p,k}(1)=\lim_{k\to1}\left[\ln(pk)-\sum_{n=0}^{p}\frac{1}{(nk+1)}\right]$$

$$= \ln p - \sum_{n=0}^{p} \frac{1}{n+1}.$$

Then,

$$\begin{aligned}
\lim_{p\to\infty}\left(\lim_{k\to 1}\gamma_{p,k}\right) &= \lim_{p\to\infty}\left(-\lim_{k\to 1}\psi_{p,k}(1)\right)\\
&= -\lim_{p\to\infty}\left[\ln p - \sum_{n=0}^{p}\frac{1}{n+1}\right]\\
&= -\lim_{p\to\infty}\left[\ln p - \sum_{n=1}^{p}\frac{1}{n} + \sum_{n=1}^{p}\frac{1}{n} - 1 - \sum_{n=1}^{p}\frac{1}{n+1}\right]\\
&= -\lim_{p\to\infty}\left[\ln p - \sum_{n=1}^{p}\frac{1}{n}\right] + 1 - \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right]\\
&= \gamma.
\end{aligned}$$

□

Definition 2.2. *A function f is said to be logarithmically convex if the following inequality holds for all $x, y > 0$.*

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y)$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^{\alpha}(f(y))^{\beta}$$

where $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.

Theorem 2.1. *The function, $\Gamma_{p,k}(x)$ is logarithmically convex.*

Proof. We want to show that for $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$,

$$\Gamma_{p,k}(\alpha x + \beta y) \leq (\Gamma_{p,k}(x))^{\alpha}(\Gamma_{p,k}(y))^{\beta}. \tag{13}$$

Recall that the Young's inequality is given by

$$x^{\alpha} y^{\beta} \leq \alpha x + \beta y \tag{14}$$

where $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. By this, we obtain

$$\left(k + \frac{x}{t}\right)^{\alpha}\left(k + \frac{y}{t}\right)^{\beta} \leq k + \frac{\alpha x + \beta y}{t}. \tag{15}$$

Next, taking $\prod_{t=1}^{p}$ on (15), we have

$$\prod_{t=1}^{p}\left(k + \frac{x}{t}\right)^{\alpha}\left(k + \frac{y}{t}\right)^{\beta} \leq \prod_{t=1}^{p}\left(k + \frac{\alpha x + \beta y}{t}\right)$$

which implies

$$\left(\frac{(x+k)(x+2k)\dots(x+pk)}{1 \times 2 \times \dots \times p}\right)^{\alpha}\left(\frac{(y+k)(y+2k)\dots(y+pk)}{1 \times 2 \times \dots \times p}\right)^{\beta}$$

$$\leq \frac{(\alpha x+\beta y+k)(\alpha x+\beta y+2k)\dots(\alpha x+\beta y+pk)}{1\times 2\times\cdots\times p}$$

which further implies

$$\frac{p!}{(\alpha x+\beta y+k)(\alpha x+\beta y+2k)\dots(\alpha x+\beta y+pk)} \leq \left(\frac{p!}{(x+k)(x+2k)\dots(x+pk)}\right)^{\alpha}\left(\frac{p!}{(y+k)(y+2k)\dots(y+pk)}\right)^{\beta}. \quad (16)$$

Then, by multiplying (16) by the identities:

$$\frac{1}{\alpha x+\beta y} \leq \frac{1}{x^{\alpha}y^{\beta}},$$
$$(p+1) = (p+1)^{\alpha+\beta},$$
$$k^{p+1} = (k^{p+1})^{\alpha+\beta},$$
$$(pk)^{\frac{\alpha x+\beta y}{k}-1} = (pk)^{\frac{\alpha x}{k}-\alpha}(pk)^{\frac{\beta y}{k}-\beta}$$

we obtain

$$\frac{(p+1)!k^{p+1}(pk)^{\frac{\alpha x+\beta y}{k}-1}}{(\alpha x+\beta y)(\alpha x+\beta y+k)(\alpha x+\beta y+2k)\dots(\alpha x+\beta y+pk)} \leq \left(\frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)}\right)^{\alpha}\left(\frac{(p+1)!k^{p+1}(pk)^{\frac{y}{k}-1}}{y(y+k)(y+2k)\dots(y+pk)}\right)^{\beta}$$

which is (13). That completes the proof. □

Remark 2.2. *Alternatively, a compact proof of Theorem 2.1 could have been as follows. By using the defintion of $\psi_{p,k}(x)$ and the fact that $\psi'_{p,k}(x) > 0$, it follows immediately that $\Gamma_{p,k}(x)$ is logarithmically convex. Then, from the definition 2.2 for $x, y > 0$, $\alpha, \beta > 0$ such that $\alpha+\beta = 1$, we obtain the inequality (13).*

Corollary 2.1. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\Gamma_{p,k}\left(\frac{x+y}{2}\right) \leq \sqrt{\Gamma_{p,k}(x)\Gamma_{p,k}(y)} \quad (17)$$

holds for $x, y > 0$.

Proof. This follows directly from Theorem 2.1 by letting $\alpha = \beta = \frac{1}{2}$. □

Theorem 2.2. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\Gamma_{p,k}(nx) \leq (pk)^{\frac{x}{k}(n-1)}\Gamma_{p,k}(x) \quad (18)$$

holds for $n \in \mathbb{N}$ and $x > 0$.

Proof. It follows easily from (2) that

$$\frac{\Gamma_{p,k}(nx)}{\Gamma_{p,k}(x)} = (pk)^{\frac{x}{k}(n-1)}\frac{x(x+k)(x+2k)\dots(x+pk)}{nx(nx+k)(nx+2k)\dots(nx+pk)}$$

which completes the proof. □

Corollary 2.2. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\Gamma_{p,k}(x+y) \leq (pk)^{\frac{x+y}{2k}} \sqrt{\Gamma_{p,k}(x)\Gamma_{p,k}(y)} \tag{19}$$

holds for $x, y > 0$.

Proof. From (17), and by using (18) for $n = 2$, we obtain

$$\begin{aligned}\Gamma_{p,k}(x+y) &\leq \sqrt{\Gamma_{p,k}(2x)\Gamma_{p,k}(2y)} \\ &\leq (pk)^{\frac{x+y}{2k}} \sqrt{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}.\end{aligned}$$

□

Remark 2.3. *Results similar to (13), (17), (18) and (19) for the (q, k)-analogue of the Gamma function can also be found in [3].*

Lemma 2.3 ([9])**.** *Let $f : (0, \infty) \to (0, \infty)$ be a differentiable, logarithmically convex function. Then the function*

$$g(x) = \frac{(f(x))^\alpha}{f(\alpha x)}, \quad \alpha \geq 1$$

is decreasing on its domain.

Theorem 2.3. *Let $p \in \mathbb{N}$, $k > 0$ and $\alpha \geq 1$. Then the inequality*

$$\frac{[\Gamma_{p,k}(y)]^\alpha}{\Gamma_{p,k}(\alpha y)} \leq \frac{[\Gamma_{p,k}(x)]^\alpha}{\Gamma_{p,k}(\alpha x)} \leq \frac{p}{p+1} k^{1-\alpha} \frac{1}{\Gamma_p(\alpha)} \tag{20}$$

is valid for $k \leq x \leq y$.

Proof. Recall from Theorem 2.1 that $\Gamma_{p,k}(x)$ is logarithmically convex. Then by Lemma 2.3, the function $H(x) = \frac{[\Gamma_{p,k}(x)]^\alpha}{\Gamma_{p,k}(\alpha x)}$ is decreasing. Hence for $k \leq x \leq y$, we have $H(y) \leq H(x) \leq H(k)$ yielding the result (20). □

Theorem 2.4. *Let $p \in \mathbb{N}$, $k > 0$ and $\alpha \geq 1$. Then the inequality*

$$\frac{[\Gamma_{p,k}(1+k)]^\alpha}{\Gamma_{p,k}(\alpha+k)} \leq \frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)} \leq 1 \tag{21}$$

is valid for $x \in [0, 1]$.

Proof. Define Q by $Q(x) = \frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)}$ for $p \in \mathbb{N}$, $k > 0$ and $\alpha \geq 1$. Let $\lambda(x) = \ln Q(x)$. Then

$$\begin{aligned}\lambda'(x) &= \alpha \frac{\Gamma'_{p,k}(x+k)}{\Gamma_{p,k}(x+k)} - \alpha \frac{\Gamma'_{p,k}(\alpha x+k)}{\Gamma_{p,k}(\alpha x+k)} \\ &= \alpha \left[\psi_{p,k}(x+k) - \psi_{p,k}(\alpha x+k)\right] \\ &\leq 0\end{aligned}$$

since $\psi_{p,k}(x)$ is increasing for $x > 0$. Hence $Q(x)$ is decreasing on $[0, \infty)$. Then for $x \in [0, 1]$, we obtain $Q(1) \leq Q(x) \leq Q(0)$ yielding the result (21). □

Remark 2.4. *By letting $p \to \infty$ as $k \to 1$ in (21), we recover the results of [10] as a special case.*

Theorem 2.5. *Let $p \in \mathbb{N}$, $k > 0$, $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$ and $m, n \in \mathbb{N}$ such that $\frac{m}{a} + \frac{n}{b} \in \mathbb{N}$. Then, the inequality*

$$\left|\psi_{p,k}^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a} + \frac{y}{b}\right)\right| \leq \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{a}} \left|\psi_{p,k}^{(n)}(y)\right|^{\frac{1}{b}} \tag{22}$$

holds for $x > 0$ and $y > 0$.

Proof. From the integral representation (9), we obtain

$$\begin{aligned}
\left|\psi_{p,k}^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a} + \frac{y}{b}\right)\right| &= \int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}}\right) t^{\frac{m}{a}+\frac{n}{b}} e^{-\left(\frac{x}{a}+\frac{y}{b}\right)t}\, dt \\
&= \int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}}\right)^{\frac{1}{a}+\frac{1}{b}} t^{\frac{m}{a}+\frac{n}{b}} e^{-\left(\frac{x}{a}+\frac{y}{b}\right)t}\, dt \\
&= \int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}}\right)^{\frac{1}{a}} t^{\frac{m}{a}} e^{-\frac{xt}{a}} . \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}}\right)^{\frac{1}{b}} t^{\frac{n}{b}} e^{-\frac{yt}{b}}\, dt \\
&\leq \left[\int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}}\right) t^m e^{-xt}\, dt\right]^{\frac{1}{a}} \\
&\quad \times \left[\int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}}\right) t^n e^{-yt}\, dt\right]^{\frac{1}{b}} \\
&= \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{a}} \left|\psi_{p,k}^{(n)}(y)\right|^{\frac{1}{b}}
\end{aligned}$$

which concludes the proof. □

Note: The absolute signs in (22) are not required if m and n are positive odd integers such that $\frac{m+1}{a}, \frac{n+1}{b} \in \mathbb{N}$.

Corollary 2.3. *Let $p \in \mathbb{N}$, $k > 0$ and $m \in \mathbb{N}$. Then the inequality*

$$\left|\psi_{p,k}^{(m)}(x)\right| \left|\psi_{p,k}^{(m+2)}(x)\right| - \left|\psi_{p,k}^{(m+1)}(x)\right|^2 \geq 0$$

holds for $x > 0$.

Proof. This follows from Theorem 2.5 by letting $x = y$, $a = b = 2$ and $n = m + 2$. □

Remark 2.5. *By letting $p \to \infty$ in Theorem 2.5, we obtain the k-analogue of (22). Also, by letting $k \to 1$ in Theorem 2.5, we obtain the p-analogue of (22) as presented in Theorem 2.1 of [5].*

Remark 2.6. *By letting $p \to \infty$ as $k \to 1$ in Theorem 2.5, we obtain Theorem 2.5 of [11] as a special case.*

Remark 2.7. *Let $x = y$ and $a = b = 2$ in Theorem 2.5. Then, by letting $p \to \infty$ as $k \to 1$, we obtain Theorem 2.1 of [4].*

Theorem 2.6. *Let $m, n, p \in \mathbb{N}$ and $k > 0$. Then, the inequality*

$$\left[\left|\psi_{p,k}^{(m)}(x)\right| + \left|\psi_{p,k}^{(n)}(y)\right|\right]^{\frac{1}{u}} \leq \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{u}} + \left|\psi_{p,k}^{(n)}(y)\right|^{\frac{1}{u}} \tag{23}$$

holds for $x > 0$ and $y > 0$, where u is a positive integer.

Proof. We employ the Minkowski's inequality for finite sums, and the fact that $a^u + b^u \leq (a+b)^u$, for $a, b \geq 0$ and u a positive integer. From (8), we obtain

$$\begin{aligned}
\left[\left|\psi_{p,k}^{(m)}(x)\right| + \left|\psi_{p,k}^{(n)}(y)\right|\right]^{\frac{1}{u}} &= \left[\sum_{i=0}^{p} \frac{m!}{(ik+x)^{m+1}} + \sum_{i=0}^{p} \frac{n!}{(ik+y)^{n+1}}\right]^{\frac{1}{u}} \\
&= \left[\sum_{i=0}^{p} \left(\left(\frac{(m!)^{\frac{1}{u}}}{(ik+x)^{\frac{m+1}{u}}} \right)^u + \left(\frac{(n!)^{\frac{1}{u}}}{(ik+y)^{\frac{n+1}{u}}} \right)^u \right)\right]^{\frac{1}{u}} \\
&\leq \left[\sum_{i=0}^{p} \left(\left(\frac{(m!)^{\frac{1}{u}}}{(ik+x)^{\frac{m+1}{u}}} \right) + \left(\frac{(n!)^{\frac{1}{u}}}{(ik+y)^{\frac{n+1}{u}}} \right) \right)^u\right]^{\frac{1}{u}} \\
&\leq \left[\sum_{i=0}^{p} \left(\frac{(m!)^{\frac{1}{u}}}{(ik+x)^{\frac{m+1}{u}}} \right)^u\right]^{\frac{1}{u}} + \left[\sum_{i=0}^{p} \left(\frac{(n!)^{\frac{1}{u}}}{(ik+y)^{\frac{n+1}{u}}} \right)^u\right]^{\frac{1}{u}} \\
&= \left|\psi_{p,k}^{(m)}(x)\right|^{\frac{1}{u}} + \left|\psi_{p,k}^{(n)}(y)\right|^{\frac{1}{u}}
\end{aligned}$$

which concludes the proof. □

Note: The absolute signs in (23) are not required if m and n are positive odd integers.

Remark 2.8. *By letting $p \to \infty$ in Theorems 2.5 and 2.6, we obtain the k-analogues of (22) and (23).*

Remark 2.9. *By letting $k \to 1$ in Theorems 2.5 and 2.6, we obtain the p-analogues of (22) and (23) as presented in [5].*

Remark 2.10. *The q-analogues, (p, q)-analogues and (q, k)-analogues of the inequalities (22) and (23) can be found in [12], [6] and [7] respectively.*

Theorem 2.7. *Let $p \in \mathbb{N}$, $k > 0$ and $m \in \mathbb{N}$. Then, the inequalities*

$$\left(\exp \psi_{p,k}^{(m)}(x)\right)^2 \geq \exp \psi_{p,k}^{(m+1)}(x). \exp \psi_{p,k}^{(m-1)}(x), \quad \textit{if } m \textit{ is odd} \tag{24}$$

$$\left(\exp \psi_{p,k}^{(m)}(x)\right)^2 \leq \exp \psi_{p,k}^{(m+1)}(x). \exp \psi_{p,k}^{(m-1)}(x), \quad \textit{if } m \textit{ is even} \tag{25}$$

are satisfied for $x > 0$.

Proof. By relation (8), we obtain

$$\psi_{p,k}^{(m)}(x) - \frac{1}{2}\left[\psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x)\right]$$

$$
\begin{aligned}
&= \sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk+x)^{m+1}} - \frac{1}{2}\sum_{n=0}^{p} \frac{(-1)^{m+2}(m+1)!}{(nk+x)^{m+2}} - \frac{1}{2}\sum_{n=0}^{p} \frac{(-1)^{m}(m-1)!}{(nk+x)^{m}} \\
&= \frac{(-1)^m}{2}\left[2\sum_{n=0}^{p} \frac{-m!}{(nk+x)^{m+1}} - \sum_{n=0}^{p} \frac{(m+1)!}{(nk+x)^{m+2}} - \sum_{n=0}^{p} \frac{(m-1)!}{(nk+x)^{m}}\right] \\
&= \frac{(-1)^{m+1}}{2}\left[\sum_{n=0}^{p} \frac{2m!}{(nk+x)^{m+1}} + \sum_{n=0}^{p} \frac{(m+1)!}{(nk+x)^{m+2}} + \sum_{n=0}^{p} \frac{(m-1)!}{(nk+x)^{m}}\right] \\
&= \frac{(-1)^{m+1}}{2}\sum_{n=0}^{p} \frac{(m-1)!}{(nk+x)^{m}}\left[\frac{2m}{nk+x} + \frac{(m+1)m}{(nk+x)^2} + 1\right] \\
&= \frac{(-1)^{m+1}}{2}\sum_{n=0}^{p} \frac{(m-1)!}{(nk+x)^{m+2}}\left[(m+nk+x)^2 + m\right] \\
&= \begin{cases} \geq 0, & m \text{ odd} \\ \leq 0, & m \text{ even.} \end{cases}
\end{aligned}
$$

That implies,

$$2\psi_{p,k}^{(m)}(x) \geq \psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x) \tag{26}$$

and

$$2\psi_{p,k}^{(m)}(x) \leq \psi_{p,k}^{(m+1)}(x) + \psi_{p,k}^{(m-1)}(x) \tag{27}$$

respectively for odd m and even m. Then, by exponentiating the inequalities (26) and (27), we obtain the desired results. □

Remark 2.11. *By letting $p \to \infty$ in Theorem 2.7, we obtain the k-analogues of (24) and (25).*

Remark 2.12. *By letting $k \to 1$ in Theorem 2.7, we obtain the p-analogues of (24) and (25) as presented in Theorem 2.5 of [5] as a special case.*

Remark 2.13. *By letting $p \to \infty$ as $k \to 1$ in Theorem 2.7, we obtain Theorem 3.2 of [8] as a special case.*

3. Conclusion

We have introduced a new two-parameter deformation of the classical Gamma function, called the (p,k)-analogue. In addition, we have established some identities and inequalities involving this new function. The established results provide generalizations of some known results in the literature.

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