UNIVERSITY FOR DEVELOPMENT STUDIES

NEW LIFETIME STATISTICAL DISTRIBUTIONS FOR SYSTEMS CONNECTED IN SERIES

BY

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DECLARATION

Student:

I declare that this thesis submitted is my original work towards the award of Doctor of Philosophy in Statistics and that no part of it has been presented for another degree in this university or elsewhere. Also it does not contain materials already published by any person except where due acknowledgement had been made in the text.

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We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision laid down by the University for Development studies.

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DEDICATION

This work is dedicated to my husband (Mr. John Abonongo), my children (Austin Awinebono Abonongo, Andrew Abowine Abonongo and Audric Awineguya Abonongo) my mum (Susana Anibatane Logubayom) and father-in-law (Mr. Lawrence Abonongo).

ABSTRACT

In probability distribution theory, substantial efforts have been made in developing probability distributions for modelling lifetime from systems connected in series. However, there are several significant situations where empirical data set from such systems do not follow any of these existing distributions. Hence, it is essential to generate more flexible distributions for modelling lifetime data from series connected components. In this study, the Nadarajah Haghighi generalised power Weibull (NHGPW) distribution and the power series generalised power Weibull (PGPW) class of distributions were developed based on the concept of compounding. From the results, the NHGPW distribution hazard function can be constant, monotonic, bathtub, unimodal, modified bathtub or modified unimodal. Its probability density function can also be bathtub, monotonic, positively skewed, unimodal, modified bathtub and modified unimodal. Monte Carlo simulations performed to assess the performance of the estimators of the NHGPW distribution showed that, its parameter estimates are consistent since their mean square error and average bias approaches zero as the sample size increase. Also, the NHGPW distribution provides a better fit to two lifetime data set than the other competitive distributions for system connected in series. The PGPW class of distribution have four sub-family of distribution; the generalised power geometric (GPG) family, generalised power poison (GPP) family, generalised power binomial (GPB) family and the generalised power logarithmic (GPL) family of distributions. The hazard rate and PDF plots of the four sub-family of distributions showed that, their hazard and PDF can be monotonic, bathtub, unimodal, modified bathtub or modified unimodal. Monte Carlo simulation performed on these sub-family of distribution showed that, their MLE estimators are consistent. The GPG family of distributions provides a better fit failure data from air conditioning system of an aircraft whiles the GPP family also provides a better fit among the four fitted distributions to service times of aircraft data. It is therefore recommended that, these new distributions be considered in modelling lifetime data from series connected systems.

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LIST OF ACRONYMS

AD	Anderson Darling
AIC	Akaike Information Criterion
AICc	Akaike Information Criterion Corrected
BIC	Bayesian Information Criterion
B.Sk	Bowley's Skewness
CDF	Cumulative Density Function
CVM	Cramér-Von Mises
GPW	Generalised Power Weibull
KS	Kolmogorov-Smirmov
Max	Maximum
Min	Minimum
M.Ku	Moor's Kurtosis
MLE	Maximum Likelihood Estimation
NH	Nadarajah Haghighi
NHGPW	Nadarajah Haghighi Generalised Power Weibull
PDF	Probability Density Function
PS	Power Series
PGPW	Power Series Generalised Power Weibull

CHAPTER 1

INTRODUCTION

1.1 Study Background

The quality of every parametric statistical analysis is determined greatly by the probability distribution assumed. For this reason, extensive efforts have been made in developing new and standard probability distributions with various statistical techniques for many lifetime situations. Nonetheless, applications from various fields such as environment, finance/economics, biological sciences, engineering, agriculture among others, have additionally shown that, many of the data sets following these traditional distributions are usually an exception rather than a certainty (Arthur et al., 2014).

In reliability and survival modelling, probability distributions are mostly used for modelling time to failure data. In many cases, the quality of the model significantly depends on; the success in selecting an appropriate probability distribution of the phenomenon under discussion; the fundamental properties of the assumed distribution; and the elementary assumptions considered in deriving these statistical distribution. These properties and assumptions significantly support in distinguishing the practical circumstances where the distributions are applicable. The more clearly determined the structural properties, the clearer the scope of the distribution. During the past decades, well known classical lifetime distributions like the exponential, Weibull, Rayleigh, linear failure rate, gamma and their extensions were used for modelling lifetime data (Lemonte, 2013). However, these traditional distributions are only applicable to lifetime time data with monotonic shape hazard rate.

In order to increase the flexibility of these well-known distributions, many researchers have proposed different transformations of them and have used these extended forms in several areas (Mahmoud, 2018). In distribution theory, generalisation of existing statistical distributions can be done by transformations, extensions or by compounding two distributions whiles introducing additional shape or scale parameter(s). Compounding two distributions can be done in three forms; by discrete-continuous compounding which involves combining a discrete and continuous distributions or discrete-discrete compounding which involves combining two discrete distributions or continuous-continuous compounding which involves combining two continuous distributions. This compounding approach can be suitable in manufacturing, biological, medical and reliability analysis for modeling failure rate data from a component/system with dual sub-systems working independently in successions or series at an expected time. For two sub-systems functioning in series, the main system fails if any one or the two sub-systems stops functioning. The stochastic representation of their failure time is given as;

$$T = min(T_1, T_2) \tag{1.1}$$

where T_1 and T_2 are the lifetime failure rate random variables for the two sub-systems. Compounding two or more distributions have been shown to be very useful in discovering various skewed and tailed properties of many distributions and for improving the goodness-of-fit of the traditional distributions (Cordeiro et al., 2017). This is done also to make the existing distributions better-off and more flexible for various lifetime situations. Most of the resulting new distributions usually contain the baseline distribution as a special case for various parameter values.

The exponential model is one of the extensively used statistical models for lifetime data analysis in reliability and survival studies. If failure rate in phenomena is constant over time, then the exponential distribution can provide an adequate fit. It however does not offer an adequate fit for failure rates which are not constant such as increasing, decreasing, bathtub and unimodal commonly encountered in biological, engineering, manufacturing and other reliability and survival analyses. Several extensions of the exponential distribution have been done, one of which is the Nadarajah-Haghighi (NH) distribution.

The NH distribution is a bi-parameter distribution proposed by Nadarajah and Haghighi (2011) as a generalisation of the one-parameter exponential model. They two researchers indicated that, their distribution has a good property of having a zero mode with increased, decreased or constant hazard rate. The NH distribution was developed as a substitute to the Weibull, gamma, exponentiated exponential and exponential distributions. The researchers provided three possible inspirations for this different family; firstly, the connection between its probability density and its hazard rate functions. The NH distribution exhibits decreasing or constant hazard rate when its corresponding probability density function is monotonically decreasing and also exhibit an increasing hazard rate for a monotonically decreasing probability density. This property is a serious limitation for the alternative distribution (thus the Weibull, gamma, and the exponentiated exponential distributions); secondly, the NH distribution always have zero chance that its shape is unimodal hence is able to model lifetime data with their zero mode. This is also a limitation for the substitute distributions; lastly, the NH distribution can similarly be inferred as a truncated Weibull distribution. Similar to the exponentiated exponential and the Weibull distributions, the NH model also has a closed form survival and hazard function. The most significant weakness of this new distribution is its failure to model hazard/failure rate that are not monotonic, for instance the bathtub-shaped, upside down bathtub (unimodal) and modified down bathtub failure rates. Therefore further extension of it is necessary.

The Weibull distribution was developed by a Swedish engineer and mathematician Weibull (1939) which was extensively further studied by him in 1951. The major advantages of the Weibull distribution are; it is able to extensively deliver practical precise failure rate analysis and projections with exceedingly smaller samples size; also, the Weibull family can give very simple and valuable plot of the failure data which are very useful in engineering studies; additionally, the Weibull distribution is very useful even with shortfalls in a data set.

When modelling monotonic failure or hazard rate functions, the Weibull model may be a good choice since its hazard can increase, or decrease or remain constant. The Weibull distribution however does not conversely give a suitable fit for data sets with bathtub or upside down bathtub shaped (unimodal) failure rates (Sujata and Rajash, 1988). Because of these limitations of the traditional Weibull distribution, many modifications of it have been proposed recently to make it more flexible in modeling data that exhibit different kinds of failure rates (for example; the modified Weibull distribution by Sarhan and Zaindin (2009); the reduced Weibull model by Almalki (2013); the generalized power

Weibull model derived by Bagdonavicius and Nikulin (2002) among others). The generalized power Weibull (GPW) model derived by Bagdonavicius and Nikulin (2002) was further studied by Nikulin and Haghighi (2006), Lai (2013) and Nikulin and Haghighi (2009) among others.

In this study, two new distributions were developed by the compounding approach. Thus; the Nadarajah Haghighi generalised power Weibull (NHGPW) by continuous-continuous compounding the NH and GPW distributions; also, by discrete-continuous compounding, the power series generalised power Weibull (PGPW) class of distributions was also developed. Various statistical properties of these distributions were derived. The parameter estimates for the new distributions were presented. Simulations analyses were also performed to evaluate the performance of the derived estimators. Each distribution developed was also applied to two lifetime data set.

1.2 Problem Statement

Usually data coming from different fields of study may exhibit different characteristics such as skewness, kurtosis and sometimes the hazard rate may exhibit different kinds of non-monotone shaped hazard rate such as bathtub, unimodal(as in engineering processes), modified bathtub and non-monotonically increasing failure rate. However, the existing distributions may not provide appropriate fit to these kinds of dataset.

In probability distribution theory, significant efforts have been made in developing new classes of standard statistical distributions for many lifetime situations. Nonetheless, there are several significant situations where empirical data set does not follow these standard and traditional statistical distributions.

Also in reliability and biological studies, a component or system may contain sub-systems connected in series with each of the sub-systems functioning independently and with their failure rate following independent distributions. For such system, the main component will fail if any one or both of the sub-systems fail. There are however limited statistical distributions developed for modelling lifetime data from such series systems. There is also a possibility that some lifetime data set obtained from such series systems might not follow any of the existing distributions. This might be due to the fact that, the time of life or failure can have different interpretations depending on the area of applications (Lai, 2013). Hence, there is the need to generate more flexible distributions for modelling the failure rate of various kinds of random variables from series connected components. This can be achieved by compounding two or more distributions. This technique allow for greater flexibility of the tails and are motivated for engineering and biological applications. Besides, compounding families might be suitable for complementary risk problems based in the presence of latent risks. The compounding techniques was pionnered by Adamidis and Loukas (1998). A numbers of researchers have developed distributions using the concept of compounding since its inception. Some of these researchers are; Barreto-Souza, (2011), Codeiro et al., (2014), Nasiru (2016), Cordeiro, (2018), Ferdnando et al. (2019) among others.

Alternatively, researches in the area of non-parametric statistics may proposed non-parametric methods for analysing data coming from such systems. However, these non-parametric methods can be computationally intensive and may also lead to loss of information and power when the parametric models are appropriate and available (Nasiru, 2018). Hence, researches in the area of distribution theory tend to extend, generalised or compound the existing parametric distributions to make them more flexible in modeling. To fill this gap, this study developed two new lifetime distribution named; the Nadrajah Haghighi generalised power Weibull (NHGPW) distribution by continuous-continuous compounding the NH and the GPW distribution; the power series generalised power Weibull (PGPW) distribution from the generalised power Weibull and the power series family. These distributions were developed on the assumption that, the fairlure rate associated with the two sub-components are independent random variables.

1.3 Objectives of Study

1.3.1 General Objective

The general objective is to develop new lifetime statistical distributions for modelling failure rate data from systems with sub-components connected in series.

1.3.2 Specific Objectives

The specific objectives are to:

- Develop the NHGPW distribution.
- Develop the power series GPW distribution.
- Derive the statistical properties of the developed lifetime distributions.
- Develop estimators for the parameters of the new distributions.
- Perform simulations analyses to assess the performance of the developed estimators.
- Demonstrate the applications of the developed distributions using lifetime data.

1.4 Significance of the study

The major motivations for introducing these new distributions are; to develop distributions that accommodate both monotonic and non-monotonic hazard shapes; to developed distribution for modelling lifetime data from series connected systems with better fits than some widely known lifetime models and other generalisations of the GPW and NH distributions. This work is also important in survival analysis and can be used in many applications in fields like biological sciences, economics, engineering, physics, social sciences, among others for modeling systems composed of two independent components in series. This study also adds to literature, distributions that can be employed in analysing data on systems connected in series. It can as well serve as a ground for further research in probability distributions.

1.5 Outline of Thesis

This thesis is categorized into six main chapters; Chapter one gives the introduction and comprises the research background, problem statement, objectives and significance of the study. Chapter two is the review of literature. Chapter three outlines the methodology employed in developing the new distributions, deriving their statistical properties, deriving estimators of parameters, goodness of fit analysis and model selection criteria. Chapter four presents the NHGPW distribution, its statistical properties, estimators of parameters, simulation analysis and applications. Chapter five presents the PGPW class of distributions, its statiscal properties, estimators of parameters, sub-families, simulation analysis and applications to lifetime data. Lastly, chapter six consist of conclusions and recommendations made from the study.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter considers relevant literature on the topic under consideration. It contains relevant information on bathtub distributions, systems connected in series and a comprehensive look at some modifications of the exponential, NH, Weibull and the GPW distributions.

2.2 Bathtub Distribution

Efforts in modelling lifetime data have been restricted to three main distributions; exponential model, decreasing failure function and increasing failure rate distributions. There seem to be Emergent interest in non-monotonic failure rate distributions; thus the bathtub models. These distributions gives acceptable models for lifetime data related to biological organisms as well as many industrial trials. (Sujata and Rajash, 1988).

Bathtub distributions are described by high failure rate value at an initial time, which subsequently decreases over time to a minimum value. From the minimum, the failure might be constant for some time and finally increase speedily. Bathtub distribution offers reasonable models in survival analysis because they can explain the complete failure rate of a system, whether biological or non-biological. Bathtub distributions can model the complete failure in the total life span of a manufactured product or biological organism. Thus it captures; the initial face of life of a product which is characterized with high failure rate; the middle face of life with nearly constant failure; as well as the advanced period with increasing failure due to actual ageing or wearing out (Sujata and Rajash, 1988).

Unimodal failures or upside-down-bathtub failure rates are also arising in many areas in reliability and biological fields. Unimodal distributions are suggested to model systems with comparatively high failure rate in the mid expected lifetime. They are normally used when the period of peaked failure is of major interest. Unimodal failure rate distributions have two shape parameters; one of which specifies the period of the modal failure and the other controls the peakness of the mode. Unimodal failures can be detected in the progress of a disease whose death rate reaches a maximum value after a fixed time and decays slowly.

2.3 Concept of Compounding Distributions

Another major technique that has been considered in developing distributions for series and parralle connected systems is the compounding approach. The compounding technique allow for greater flexibility of the tails of a distribution and are applicable in engineering and biological fields. It is also very useful when dealing with complementary risk problems based in the presence of latent risks. The compounding approach was proposed by Adamidis and Loukas (1998), where they introduced the exponential geometric distribution by taking the minimum of the two distributions. Since its inception, many researchers have proposed new distributions using this concept.

Compounding of distributions can be done by three major approaches; combining two continuous distributions (continuous-continuous compounding), or combining two discrete distributions (discrete-discrete compounding) or combining a continuous and a discrete distribution (continuous-discrete compounding or discrete-continuous compounding). The concept of compounding is based on a system or component containing two independent sub-components either connected in series or connected in parallel with the failure rate distribution of each sub-components following two different distributions. In these situations, attempts are made to derive a single distribution that models the failure rate of the main system based on the failure rate distributions of the sub-components.

In reliability and biological studies, series systems are common system configuration. If two independent components are connected in series, then the main component fails if one or both sub-components fails. Several researchers have developed statistical distributions for modeling situations of this kind. Some of these are;

Nasiru (2016) developed the serial Weibull Rayleigh distribution by combining the Rayleigh and Weibull distributions. His distribution contains the Weibull, Rayleigh, exponential and the linear failure rate distributions as sub-distributions. Fernando, (2017) developed the Weibull NH distribution by compounding the NH and Weibull distributions. Fernando et al. (2019) developed the Nadarajah-Haghighi-Lindley distribution by continuous-continuous compounding the NH and Lindley distributions. Cicero et al. (2000) developed the beta NH distribution by combining the beta distribution and NH distributions. Cordeiro, (2014) developed the exponential-Weibull lifetime distribution for modeling system with serial connection. Cordeiro et al. (2018) developed the Lindley-Weibull distribution by joining the Lindley and Weibull distributions. Ortega et al. (2015) derived the poisson-gamma NH distribution.

Marinho (2016) combined the geometric and NH distributions to obtain the geometric-NH distribution. Stacy (1962) obtained the gamma-Weibull model. Gamma-modified-Weibull distribution was derived by Cordeiro et al. (2015). Bourguignon et al. (2015) compounded the Nadarajah Haghighi and gamma distributions into the gamma NH distribution. The exponential Poisson was also proposed by Kus, (2007).

Other compounded distributions developed in literature are; the Weibull geometric (Barreto-Souza et al., 2011), Pareto Poisson-Lindley (Asgharzadeh et al., 2013),the exponential-Weibull lifetime distribution by Cordeiro et al. (2014), Asgharzadeh et al. (2016) introduced the Weibull Lindley (WL) distribution among others.

2.4 Review of the Weibull Model

Due to the limitations of the traditional Weibull model, many alterations of it which resulted in new classes of distributions have been developed. Some of these modifications are; The inverse Weibull (IW) distribution was studied by Keller et al. (1984). Felipe et al. (2005) further generalised the IW distribution into the generalised IW distribution. The generalised IW as proposed has decreasing and unimodal failure rates. The reflected-Weibull distribution was also suggested by Cohen (1973) which contain the Weibull distributed as sub-model. Stacy (1962) combined some features of the gamma and Weibull distributions to derive the gamma Weibull model. Its failure rate was shown to be bathtub shape, decreasing, increasing, and unimodal.

The Kies-Phani modified Weibull distributions was proposed by Kies (1958) by adding limits to the Weibull distribution. This was to address some limitations of the application of the Weibull distribution in some area of material science like strength of brittle materials where the strength values are limited. The generalised Weibull model was derived by Mudholkar and Kollia (1994) by adding another parameter to the CDF of the Weibull model. The hazard function of this distribution are bathtub and unimodal shaped. shapes.

Xie and Lai (1996) suggested the additive Weibull model by summing the hazard functions of the two Weibull family; thus increasing and decreasing hazard rate functions. This distribution was shown to exhibit bathtub shaped hazard function. Arthur et al. (2014) further studied the additive Weibull distribution by deriving its complete moments, incomplete moments, quantile function and moments generating function. They further estimated its parameters by the maximum likelihood estimation. Ibrahim and Gokarna (2013) proposed another extension of the additive Weibull distribution titled the transmuted-additive-Weibull.

Zhang and Xie (2007) also developed the extended-Weibull distribution using the Marshall and Olkin (1997) family of distribution. A modified-Weibull model was suggested by Sarhan and Zaindin (2009) which generalised the Rayleigh, exponential, linear failure rate and the Weibull distributions. The researchers studied some statistical properties of this distribution and estimated its parameters by maximum likelihood estimation approach. The probability density function of this was shown to be decreasing or unimodal whiles its hazard rate was increasing, decreasing or bathtub shaped.

Other recent extensions of the Weibull distribution are; Kumaraswamy log-logistic Weibull distribution by Mdlongwa et al. (2019), Kumaraswamy Weibull distribution proposed by Cordeiro et al. (2010), Kumaraswamy modified Weibull distribution by Cordeiro et al. (2012), Log-Logistic Weibull distribution and its extension gamma log-logistic model by Oluyede et al. (2016) and Foya et al. (2017) respectively, The transmuted Weibull Lomax distribution by Fify et al. (2015), Marshal Olkin extended Weibull family of distributions by Santos-Nero et al. (2014), Marshal Olkin additive Weibull distribution by Ahmed et al. (2015) among others.

The generalised power Weibull (GPW) distribution by Flores et al. (2013) is another modification of the Weibull distribution which has not yet received much extension.

2.5 Review of Generalised Power Weibull Distribution

The GPW distribution was developed by Bagdonavicius and Nikulin (2002) for building accelerated failure time models to investigate the dependence of a lifetime distribution on prognostic variables (Nikulin and Haghighi, 2006). Nikulin and Haghighi (2006) showed that, the hazard rate of the GPW model can be constant, monotonic and non-monotonic shaped. Chi-square goodness test performed showed that, the GPW offers a good fit to randomly censored data. Lai (2013) described the GPW distribution as one of the generalisations of the Weibull model which is mostly essential to describe the non-monotonic nature of the observed hazard rates.

Based on the concept of exponentiated distributions, Fernando et al. (2018) introduced the four parameter exponentiated generalised power Weibull (EGPW) distribution by taking the GPW model as a parent model in the exponentiated family. The major motivations of this distribution as pointed out by Fernando et al. (2018) includes; the distribution is flexible since it has some known life distributions in-build in it. Hence can be applicable when modelling the maximum life of a sample following the GPW distribution for a parallel system in which the system works if one or more of the sub-components work. The EGPW probability density function (PDF) was also shown to be log-convex and log-concave.

Mahmoud and Abdullah (2016) derived the Kumaraswamy generalised power Weibull distribution. This family has sub models such as the Kumaraswamy Weibull, Weibull, GPW, exponentiated Weibull, and some new model such as Kumaraswamy generalized power, EGPW, exponential distributions as special cases. The parameters of this new model were obtained by the maximum likelihood estimation approach. The hazard rate function of this distribution was demonstrated to be constant, decreasing, increasing, bathtub and upside down bathtub.

Ehab and Hisham (2018) compared between the MLE and Bayesian estimators for the shape values of GPW distribution. Their application was centered on complete censoring, type II censoring and type II progressive censoring data.

2.6 Review of Exponential and the Nadarajah Haghighi Distributions

The exponential model is a well-known distribution due to its fixed hazard function and memory less feature. Over the years, the exponential model has been seen as the most important one parameter family partially based on the fact that, majority of the frequently used lifetime distributions are extension/generalizations of the exponential model. To make this distribution flexible, a good number of generalisations of it have been done. Some of these modifications are;

Gupta and Kundu (2001) proposed the exponentiated exponential (EE) distribution as a substitute to the gamma model. Ibrahim et al. (2018) derived a Kumaraswamy extension exponential distribution constructed from the Kumaraswamy family. This distribution contains the extension exponential distribution and the Kumaraswamy generalised exponential distribution as special sub-models. They derived some properties and discussed the estimators of the parameters of the new distribution.

Another important extension of the exponential model which has received much attention is the NH distribution by Nadarajah and Haghighi (2011) . This distribution was primarily developed as a substitute to the exponentiated exponential, gamma and Weibull distributions. The later distributions are flexible distributions with monotonically decreasing, unimodal density functions and monotone hazard rate function. They however do not permit increasing hazard function with corresponding decreasing probability density function as compared to the Nadarajah Haghighi distribution. The NH distribution has been extensively studied by many researchers. Among these are;

Fernando et al. (2019) developed the Nadarajah-Haghighi-Lindley (NHL) distribution by combining the Lindley and NH distributions. The main motivation of this distribution as pointed out by the researchers was its usefulness in industrial and reliability analysis for analysing data on a system with dual sub-systems operating independently in series at an assumed time. They pointed out that, this distribution accumulates the advantages of the NH distribution since the NH distribution is its sub-distribution. The NHL distribution can exhibit both monotone and non-monotone shaped hazard functions. Stenio (2015) proposed the Kumaraswamy Nadarajah-Haghighi distribution as a generalisation of the NH model. The researcher demonstrated that, the new model was moderately flexible for analysing positive data and have monotone and non-monotone hazard function conditional on the parameter values. This model contains the NH and exponentiated NH distributions as special case (Lemonte, 2013).

Lemonte (2013) also derived the exponentiated Nadarajah-Haghighi (ENH) by raising the baseline distribution function of the NH distribution to positive integer. Abdus et al. (2019) further introduced and studied the beta exponentiated NH model, whose failure rates were shown to be monotonically increasing, decreasing and non-monotonic. The researcher derived some empirical properties of this model and performed Monte Carlo simulation on the estimates of its parameters. They also defined a regression model based on the new distribution.

Cicero et al. (2000) developed the beta Nadarajah-Haghighi distribution for modelling survival data from the beta generated family. The technique of MLE was considered in estimating the parameters of the model and Monte Carlo simulation was also shown.

Marcelo et al. (2015) introduced the gamma Nadarajah Haghighi distribution also referred to as the truncated generalised gamma distribution (Stacy, 1962). Its hazard rate function was shown to be both monotonic and non-monotonic subject to the values of the parameter. Vedo et al. (2016) obtained the exponentiated generalised Nadarajah-Haghighi distribution and studied some computational and theoretical properties of it. This distribution contains the exponential, exponentiated exponential, NH, exponential and exponentiated NH distributions as sub-distributions. The also obtained some statistical properties of it and determined its parameters by maximum likelihood approach and further performed Monte Carlo simulation. This distribution was shown to be very flexible in describing complex positive real data.

Tahir et al. (2018) suggested the inverted Nadarajah-Haghighi distribution. This distribution was shown to have non-increasing and unimodal (right-skewed) density with decreasing and unimodal hazard function. The researchers further obtained some estimators (based on the frequentist and Bayes approach) of the unidentified parameters. Fernando et al. (2017) also derived the Weibull Nadarajah-Haghighi distribution by putting the NH model into the Weibull-G family by Bourguignon et al. (2015). This proposed distribution has monotonic and non-monotonic failure shapes thus overcoming the limitations of the traditional Weibull and NH distributions. The researcher explored various essential properties of the derived distribution. Its density was shown to be unimodal and is relatively flexible for data set that exhibits skewness and kurtosis.

Lima (2015) studied the Kumaraswamy Nadarajah Haghighi (KNH). A three-parameter Logistic Nadarajah Haghighi (LNH) distribution was also introduced by Fernando et al. (2017) by inputting the NH distribution into the logistic-X family pioneered by Tahir et al. (2016a). The LNH distribution was shown to have an upside-down-bathtub density. It also had high flexibility than the parent model since it tolerates monotone and non-monotone hazard forms. The new distribution was compared with the Weibull, exponential Weibull, PGW and NH distributions by some goodness-of-fit measures (Kolmogorov Smirrov and Anderson-Darling) and the researcher concluded that, the current model fit well than the compared distributions for the two data sets.

Fernando et al. (2017) in addition proposed the beta Nadarajah Haghighi distribution which have the generalised exponential distribution by (Gupta and Kundu, 2007), exponential, NH, beta exponential and the exponentiated NH (Lemonte, 2013) distributions as inbuilt models. The hazard rate function of the beta Nadarajah Haghighi distribution can have increasing, decreasing, unimodal or bathtub-shaped.

The Marshall-Olkin Nadarajah Haghighi distribution was derived by Hilary et al. (2018) using the Marshall Olkin generator by Marshall and Olkin (1997). In their paper, they obtained the ordinary differential equations of the probability function of the distribution by differentiation.

Other generalisations of the NH model proposed in literature are Poisson gamma Nadarajah-Haghighi, the transmuted Nadarajah-Haghighi, modified Nadarajah-Haghighi (MNH) among others.

2.7 Review of the Power Series Class of Distributions

The power series class is a technique of deriving new distributions. Several distributions have been derived using the power series approach. Some of these are;

The modified power series distribution was derived by Gupta (1974) for studying Lagrangian distributions. He further studied some structural properties of it. He found the recurrence relation between the central and its factorial moments. The negative moments of this distribution was studied by Kumar and Consul (1979) which can be used to find the precise quantity of bias of the MLE of modified power series distribution. Tripathi et al. (1986) derived the incomplete moments and the recurrence relation among the incomplete moments about origin of this distribution. Jani and Shah (1979b) obtained its integral function for the tail probabilities for absolutely continuous distributions. Gupta and Kundu (1982) further obtained the probability generating function of this distribution whiles Shanmugam (2001) presented the product moment generating function for this distribution. Gupta and Singh (1981) also considered the moments and factorial moments of this distribution. Shoukri and Consul (1982) developed the bivariate modified power series distribution and studied its properties.

Chahkandi and Ganjali (2009) suggested the exponential power series family which generalises the two-parameter exponential power series termed the Weibull power series class of distributions by Morais and Barreto (2011). Eisa and Mitra (2012) presented the exponentiated Weibull power series class of distributions which was gotten by compounding the exponentiated Weibull and power series distributions. The Weibull power series class of distribution can have an increasing, decreasing, and upside-down bathtub failure rate function.

Jose et al. (2013) presented the complementary exponential power series distribution with increasing failure rate which was introduced as a supplement to the exponential power series distribution proposed by Chahkandi and Ganjali (2009). Bourguignon et al. (2015) proposed a new class of fatigue life distribution known as the Birnbaum-Saunders power series class of distributions.

Said (2015) presented the generalised extended Weibull power series family of distributions which generalises the generalised power series exponential and the extended Weibull power series class distributions introduced by Mahmoudi and Jafari (2011) and Silva et al. (2013) correspondingly. This involved compounding the generalised extended Weibull distributions and power series distributions. Fernando et al. (2018) introduced the four parameter exponentiated GPW (EPGW) distribution using the exponentiated family.

Baitshephi et al. (2019) suggested the Webull-G Power Series family of distributions and its sub-model called the Weibull-G logarithmic distribution. Structural properties of this family of distributions and its sub-model were obtained. Simulation analysis to examine the bias and mean square error of the estimators for each parameter were presented.

Other application of the power series in developing distributions are; Ali et al. (2015) derived the bivariate generalised exponential power series class of distributions, the generalised exponential power series distribution by Mahmoudi and Jafari (2012), generalised linear failure rate power series distribution by Alamatsaz and Shams (2016), double bounded kumaraswamy power series class of distributions by Bidran and Nekoukhous (2013), Burr XII power series distribution by Silva and Cordeiro (2015), Lindly power series class of distributions by Gayan and P. (2015), bivariate weibull power series class of distributions by Nadarajah and R (2017) among others.

2.8 Conclusion

From the reviews, it could be seen that, the exponential and Weibull distribution are important distributions in analysis of lifetime data relating to human life, manufactured products life and a wide variety of data in survival studies. Therefore, this research developed two new distribution from the NH and the GPW distributions which are extensions of the exponential and Weibull distributions. These are; the NHGPW distribution by compounding these two continuous distributions in the concept of systems connected in series; and the power series generalised power Weibull class of distributions. various statistical properties of these distribution were developed. Adequacy and flexibility of the new proposed distributions were also tested. The parameter estimates of the distributions were developed by the maximum likelihood estimation, ordinary least square estimation and the Cramer Von Mises estimation.

CHAPTER 3

METHODOLOGY

3.1 Introduction

This chapter presents various methodologies that were employed in this study to achieve the stated objectives. It contains information on the GPW distribution, the NH distribution, maximum likelihood estimation technique, model selection criteria and goodness-fit-analysis among others.

3.2 The Nadarajah-Haghighi Distribution

The NH distribution is a generalisation of the exponential model. If a random variable T follows the NH distribution ($T \sim NH(\alpha, \beta)$), then its cumulative distribution function (CDF) is given as;

$$F(t) = 1 - e^{(1-(1+\alpha t)^{\beta})} \quad t > 0, \alpha > 0, \beta > 0. \tag{3.1}$$

With probability density, survival (reliability), hazard/failure rate and quantile functions given respectively as;

$$f(t) = \alpha\beta(1 + \alpha t)^{\beta-1} e^{(1-(1+\alpha t)^{\beta})} \quad t > 0, \tag{3.2}$$

$$s(t) = e^{(1-(1+\alpha t)^{\beta})}, \quad S(t) = [0, 1], \tag{3.3}$$

$$h(t) = \alpha\beta(1 + \alpha t)^{\beta-1}, \quad t > 0, \tag{3.4}$$

$$Q(p) = \frac{1}{\alpha}((1 - log(1 - p)^{\frac{1}{\beta}}) - 1), \quad p\epsilon[0, 1]. \tag{3.5}$$

Where β is a shape/tilt parameter and α is the scale parameter.

If $\beta = 1$, the NH distribution reduces to the exponential distribution. For larger values of α, there is a quicker decay of the higher tail.

Some attractive features of the Nadarajah Haghighi distribution as pointed out by Nadarajah and Haghighi (2011) are;

- it always have a zero mode; thus the NH distribution has a zero likelihood of its shape being unimodal hence can be used to model data set with its mode fixed at zero.
- it's failure function can be monotonically increasing, decreasing or constant.
- it permits decreasing or constant failure rate function for a corresponding monotone decreasing probability density function and increasing rate for a respective monotonically decreasing probability density function.
- the NH distribution can also be interpreted as a truncated Weibull distribution.
- the NH model has closed-form reliability and a hazard function.

3.3 The Generalised Power Weibull Distribution

The generalised Power Weibull Distribution (GPW) was suggested by Bagdonavicius and Nikulin (2002). Assuming T follows the GPW distribution ($T \sim GPW(\gamma, \theta, \lambda)$), then the cumulative distribution function of the T is;

$$F(t) = 1 - e^{(1-(1+\lambda t^{\gamma})^{\theta})}, \quad t > 0, \gamma > 0, \theta > 0, \lambda > 0. \tag{3.6}$$

With probability density, survival and hazard rate functions given as;

$$f(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}, \quad t > 0, \tag{3.7}$$

$$s(t) = e^{(1-(1+\lambda t^{\gamma})^{\theta})}, \quad s(t) = [0,1], \tag{3.8}$$

$$h(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}, \quad t > 0. \tag{3.9}$$

Where λ is the scale parameter and γ, θ are the shape parameters. For $\theta = 1$, the GPW distribution reduces to the 2-parameter Weibull distribution. if $\theta = 1, \gamma = 1$, it reduces to the exponential distribution. For $\gamma = 1$, we have the NH distribution.

3.4 The Power Series Family of Distribution

Assuming N is the number of subcomponents of a main system operating independently at a given time period, then the zero-truncated power series (PS) family have probability mass function (PMF) given as;

$$P(N = n) = \frac{a_n \alpha^n}{C(\alpha)}, n = 1, 2, ... \tag{3.10}$$

$$C(\alpha) = \sum_{i=1}^{\infty} a_n \alpha^n. \tag{3.11}$$

$a_n > 0$, $\alpha \in (0, s)$, a_n is the coefficient of the power series, $C(\alpha)$ is the generating function, s is the parameter space. The PS family are; binomial (Bin), poisson (Poi), geometric (Geo) and logarithmic (Log) distributions. Some useful quantities of this family are;

Table 3.1: **Power Series Family**

Dis	a_n	C(α)	$C'(\alpha)$	$C''(\alpha)$	$C'''(\alpha)$	s	C^{-1}	α
Geo	1	$\alpha(1-\alpha)^{-1}$	$(1-\alpha)^{-2}$	$2(1-\alpha)^{-3}$	$6(1-\alpha)^{-4}$	1	$\alpha(\alpha+1)^{-1}$	$(-\infty, 1)$
Poi	$\frac{1}{n!}$	$e^{\alpha}-1$	e^{α}	e^{α}	e^{α}	∞	$\log(\alpha+1)$	$(0, \infty)$
Log	n^{-1}	$-\log(1-\alpha)$	$(1-\alpha)^{-1}$	$(1-\alpha)^{-2}$	$2(1-\alpha)^{-3}$	1	$1-e^{-\alpha}$	$(-\infty, 1)$
Bin	$\binom{M}{n}$	$(1-\alpha)^m-1$	$\frac{m}{(1-\alpha)^{1-m}}$	$\frac{m(m-1)}{(1-\alpha)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1-\alpha)^{3-m}}$	∞	$(\alpha-1)^{\frac{1}{m}}-1$	$(0, \infty)$

3.5 Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) is used in statistics for determining the parameters of a statistical model. This approach is grounded on the likelihood function $L(\theta, T)$ of the given statistical model and finds the parameter estimates by determining the values of the parameters that maximise $L(\theta, T)$. Assuming we have a set of n measured values of a random variable given as $T = (T_1, T_2, ..., T_n)$ selected based on a family of probabilities with $f(t_i; \theta)$ as their marginal density function, then MLE finds the value of the model parameter θ, that maximises $L(\theta, T)$. Thus we select the θ values that makes the data most likely.

The likelihood function is the joint density function expressed as a function of θ. For independent measurements, the joint density function $L(\theta, T)$ is the product of the distinct/marginal densities $f(t_i;\theta)$ given as;

$$L(\theta;T) = f(t_1;\theta) \times f(t_2;\theta) \times .... \times f(t_n;\theta) = \prod_{i=1}^{n} f(t_i;\theta). \tag{3.12}$$

The estimate $\hat{\theta}$ for the parameter θ is the value which maximizes $L(\theta;T)$. This can be represented as; $\hat{\theta} = argmaxL(\theta, T)$ In practice, the MLE is obtained by maximising the score function, $lnL(\theta, T)$ or average log-likelihood, $\frac{lnL(\theta,T)}{n}$

Obtaining the ML estimates involves the following steps;

- Obtain the likelihood function of the given density function.
- Take natural log of this likelihood function to obtain the score function.
- Take partial deferential of Log-Likelihood, thus, $\frac{d}{d\theta}lnL(\theta, T)$.
- Equate the derivative to zero and solve for the parameter needed, thus $\frac{d}{d\theta}lnL(\theta, T) = 0$

3.5.1 Desirable Properties of Maximum Likelihood Estimation

The MLE technique has some desirable properties which makes it very useful in estimating parameters under various regularity conditions. These regulations are;

- the distinct pdfs have a common support for all θ_i
- the random variables have distinct pdfs such that for $\theta_i \neq \theta_j$, hence $f(t;\theta_i) \neq f(t;\theta_j)$
- the true parameter exist within an interior point in θ.

Some of the properties of the MLE technique are;

3.5.1.1 Asymptotic Consistency

The estimate $\hat{\theta}$ of the MLE is asymptotically consistent (thus as $n \longrightarrow \infty$, $\hat{\theta} \longrightarrow \theta$) for limited values of n. Since the MLE is consistent, the bias $B(\hat{\theta})$ will approach zero as the sample size (n) increases (that is as $n \longrightarrow \infty$, $B(\hat{\theta}) \longrightarrow 0$). For consistent estimators, the distribution of the estimators becomes concentrated near the true value being estimated hence the variation between the estimate and the parameter will be small and will be approaching zero as your sample size increases.

If $T = (T_1, T_2, ..., T_n)$ are random variables and $\hat{\theta}$ is an estimator, then $\hat{\theta}$ is a consistent estimator of θ iff $\hat{\theta}$ converges in probability to θ ($\hat{\theta} \longrightarrow \theta$ as $n \longrightarrow \infty$). Thus; $P(\mid \hat{\theta} - \theta \mid > \epsilon) \longrightarrow 0$ and $P(\hat{\theta} = \theta) \longrightarrow 1$ as n increases.

For the mean square errors of the estimator, $\hat{\theta}$ is consistent if the mean square error of $\hat{\theta}$ approaches zero as the sample size increases. If $\hat{\theta}$ is an unbiased estimator and the variance of $\hat{\theta}$ exist, then $\hat{\theta}$ is a consistent estimator of θ iff;

$$\lim_{n \longrightarrow \infty} V(\hat{\theta}) = \lim_{n \longrightarrow \infty} E((\hat{\theta} - \theta)^2) = 0. \tag{3.13}$$

3.5.1.2 Asymptotic Normality Property

The MLE of θ is asymptotic multivariate normal distributed with minimal variance ($var(\hat{\theta})$) under very broad conditions. $var(\hat{\theta})$ is the variance-covariance matrix associated with $\hat{\theta}$ and is given as the inverse of the fishers information ($I(\theta)$. Assuming $T = (T_1, T_2, ..., T_n)$ are sequence of independently and identically distributed observed random variables with density $f(\theta; t)$, if $\hat{\theta}$ is a MLE of θ, then;

$$\sqrt{n}(\hat{\theta} - \theta) \longrightarrow N(\theta, \frac{1}{I(\theta)}) = \sqrt{n}(\hat{\theta} - \theta) \longrightarrow N(\theta, \sigma^2). \tag{3.14}$$

where σ^2 denotes the asymptotic variance-covariance of $\hat{\theta}$.

3.5.1.3 Asymptotic Efficiency

When a number of unbiased estimators are being compared, the efficient estimator is the estimator with the smallest variance. The maximum likelihood estimates has the smallest asymptotic variance when compared with other estimators hence it is asymptotically efficient and asymptotically optimal. When $n \longrightarrow \infty$, MLE generates unbiased estimators with smallest variance. An estimator is efficient if it achieves the Cramer-Rao lower bound inequality. That is;

$$var(\hat{\theta}(t)) \geq \frac{[d/d\theta E(\hat{\theta}(t))]^2}{I(\theta)}. \tag{3.15}$$

where $I(\theta)$ is the Fisher information that measures the information carried by the observable random variable I about the unknown parameter θ for an unbiased estimator $\hat{\theta}(t)$.

$$var(\hat{\theta}(t)) \geq \frac{1}{I(\theta)} \; or \; \frac{1}{-nI(\theta)}. \tag{3.16}$$

This means that, the variance of an unbiased estimator is as least the reverse of the Fisher's information. The variance of estimator $\hat{\theta}(t)$ cannot be lower than the CRLB, any estimator whose variance is equal to the lower bound is considered as an efficient estimator or attains the CRLB.

3.5.1.4 Property of invariance

The maximum likelihood estimators are invariant under change of parameter. Thus MLE is functional under any transformation. If $L(\theta; T)$ is the likelihood function associated with a given random variable and $\hat{\theta}$ is the MLE of θ, then θ is still the MLE of $lnL(\theta; T)$. Also assuming $\hat{\theta}$ is a MLE of θ and $f(\theta)$ is a differential function, then $f(\hat{\theta})$ is the MLE of $f(\theta)$.

3.6 Model Selection Criteria

To identify the best model among candidate models, it is important to use model selection criteria for comparison. In this study, the Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICc) and the Bayesian Information Criterion (BIC) were employed to check fitness level of the developed distributions as compared to existing classical distributions.

The AIC was derived by Akaike (1974). AIC is a proposed measure of the comparative data lost for a given model. Although the AIC is able to penalise models with many parameters, it is effective when the sample size is large. The AIC is not also unbiased. Due to these limitations, the AIC corrected (AICc) was developed. The AICc is a revolution of the AIC by Hurvich and Tsai (1989) and is utilized when there is a considerable likelihood that AIC will choose models that have an excessive number of parameters (that is, AIC will over fit).

The Bayesian Information Criterion (BIC) was also proposed by Schwarz (1978). The BIC is asymptotic result inferred under the assumptions that the information dispersion is an exponential family. The BIC criterion is a consistent estimator and has the tendency to choose models with less parameter than the AIC and AICc. Hence the BIC has the power to penalise models with many parameters than the AIC and AICc in both larger and smaller samples.

For a number of competing models for a data set, the model with the minimum or most minimum values of these criteria (AIC, AICc and BIC) is the best and adequate model. These criteria will permit the individual find suitable model that best fit or clarifies the information with the base of free parameters. Both AIC and BIC determines the best model by presenting a punishment factor for the number of parameters in the model. The criteria are given as follows;

$$AIC = 2K - 2\log(L). \tag{3.17}$$

$$AIC_c = AIC + \frac{2K^2 + 2K}{n - K - 1}. \tag{3.18}$$

$$BIC = \log(n)K - 2\log(L). \tag{3.19}$$

Where L is the maximum value of likelihood function of the model, n is the number of data points or observation and k indicates the number of parameters estimated by the statistical model. The first part of these criteria is the penalty term of the criteria which penalizes a candidate model for the number of parameters used whiles the second part measures the goodness-of-fit of the statistical model to the data . Based on this penalty term, the BIC has a stiffer penalty term hence turns to select models with fewer parameters for large sample size than the AIC.

3.7 Total Time on Test

Barlow and Doksum (1972) proposed the total time test (TTT) plot and the scaled TTT transform as a tool for model identification based on data representativeness. The TTT is a graphical procedure for checking the shape of the failure/hazard rate of a given data set. By comparing the TTT plot for a given data set based on the time to failure with the different scaled TTT transform, it is possible to select a suitable lifetime distribution. Earlier application of this technique was by Aarset (1987) to investigate if a random sample comes from a bathtub distribution.

Assuming we have complete ordered sample, $T_{1;n}, T_{2;n}, ..., T_{n;n}$ of failure times from n identically and independently distributed, the TTT statistics is given as;

$$TTT_{n,i} = \sum_{j=1}^{i}(n-j+1)(t_{j;n}...t_{j-1;n}) \;\; i = 1, 2, ..., n. \tag{3.20}$$

with a scaled TTT given as;

$$TTT_i^* = \frac{TTT_{n,i}}{TTT_{n,n}} \;\; ,0 \leq TTT_{n,n} \leq 1. \tag{3.21}$$

The TTT-transform curve is obtained by plotting $1/n$ against TT_i^* in the case of complete data and $1/m$ against $TT_i^{'}$ when dealing with incomplete data where the test ends at m.

$$TT_i^{'} = \frac{TTT_{n,i}}{TTT_{n,m}} \;\; ,0 \leq TTT_{n,m} \leq 1. \tag{3.22}$$

If $s(t)$ is the survival function, then the TTT-transform is given as;

$$h^{-1}(u) = \int_0^{F-1(u)} s(t)dt, \;\; u\epsilon[0, 1]. \tag{3.23}$$

The scaled TTT is calculate as;

$$F(u) = \frac{h^{-1}(u)}{h^{-1}(1)} = \frac{\int_0^{F-1(u)} s(t)dt}{\int_0^{F-1(1)} s(t)dt}. \tag{3.24}$$

The curve of F(u) versus $\theta \leq p \leq 1$ is the scaled TTT-transformed curve. The shape of this curve can be increasing, decreasing, constant, bathtub or unimodal. Based on nature of the scaled TTT-transform curve, the shape of the hazard function of a distribution is described as;

i. Monotonically increasing if the scaled TTT-transform curve is concave above the 45^o line.

ii. Monotonically decreasing if the scaled TTT-transform curve is convex beneath the 45^o line.

iii. Bathtub shape if the scaled TTT-transform curve is first convex beneath and then concave above the line.

iv. Upside down bathtub or unimodal if the scaled TTT-transform curve is first concave above 45^o line and then convex beneath the line.

3.8 Goodness of fit Analysis

Goodness-of-fit tests are techniques used to determine whether or not a random sample comes from a hypothesized distribution. This study employed the Kolmogorov-Smirnov (KS), Crame$'$r-Von Misses (CVM) and the Anderson-Darling (AD) goodness of fit tests.

3.8.1 Kolmogorov-Smirnov test

The Kolmogorov-Smirnov statistics proposed by Kolmogorov (1933) is the best known empirical goodness of fit test for checking if the distribution of a random sample $T_1, T_2, ..., T_n$ follows a specified/hypothesised distribution. Assume $P(t_i)$ is an empirical distribution function drawn from a specified population, the Kolmogorov-Smirnov statistics investigate the hypothesis;

$H_o : F(t_i) = F^*(t_i), \;\; \infty \leq T \leq \infty$ thus the sample comes from $F^*(t_i)$

$H_1 : F(t_i) \neq F^*(t_i), \;\; \infty \leq T \leq \infty$ thus the sample does not comes from $F^*(t_i)$

$F(t_i)$ is the unknown distribution and $F^*(t_i)$ is the hypothesised distribution function which is the expected CDF of the distribution considered. The KS statistics checks this hypothesis by comparing the empirical distribution with the hypothesised distribution to see if the two agree. The KS statistics measure the largest vertical distance between $P(t_i)$ and $F(t_i)$ and is given as;

$$KS = max_t[|F^*(t_i) - P(t_i)\ \ or P(t_i) - F^*(t_i)|]. \tag{3.25}$$

$P(t_i)$ is an estimate of $F(t_i)$ and is given as;

$$P(t_i) = \frac{1}{n}\sum_{i=1}^{n} I_{t_i \leq t}. \tag{3.26}$$

$$= \frac{number\ \ of\ \ observations\ \ below\ \ t}{number\ \ of\ \ observartions}.$$

If the observations are ordered, $T_1 \leq T_2 \leq, ..., \leq T_n$, then;

$$P(t_i) = \frac{i}{n} \tag{3.27}$$

According to Law and Kelton (2000)), the KS statistic can also be determined by;

$$K^+ = max\left[\left|\frac{i}{n} - F^*(t_i)\right|\right]. \tag{3.28}$$

$$K^- = max\left[\left|F^*(t_i) - \frac{i-1}{n}\right|\right]. \tag{3.29}$$

$$KS = max[K^+, K^-]. \tag{3.30}$$

If the KS statistics exceeds the $1 - \alpha$ quantile from the KS table, thus if the KS statistic is larger than normally expected for a given sample, we reject H_o hence the theoretical distribution is not acceptable for modelling the population considered. Or if the p-value associated with the KS statistics is less than the α-level selected, we reject H_o. If more than one distributions are being compared, the distribution with the smallest KS value is the best distribution. Thus the distribution with the highest chance of accepting H_o is the best among the candidate distribution.

The KS goodness-of-fit test is mostly more powerful test of H_o than the chi-square test since with the KS test, it is not necessary to divide the observations into intervals in situations where the hypothesised distribution is continuous. Hence the problem associated with small expected frequencies and smaller number of intervals will not be encountered as can be in chi-square test.

The KS test involves the following steps;

i. Order the data from the smallest to the highest

ii. Compute $P(t_i)$ for each observation

iii. For each observation, determine $F^*(t_i)$ using the table of the distribution you expect your random sample to follow

iv. Compute the absolute difference between the entries in the table of observation and the table of expected values

v. Obtain the KS statistics

vi. Compare the KS statistics with the KS table value

3.8.2 Cramér-Von Misses Test

This test was proposed by Cramer (1928). It is also base on the empirical distribution function and it gives on the square integral of the discrepancies between the hypothesised CDF and the empirical distribution function.

$$C^* = \int_{-\infty}^{\infty} \{F^*(t_i) - P(t_i)\}^2 w(t)dt = \int_{-\infty}^{\infty} \{F^*(t_i) - P(t_i)\}^2 dF^*(t_i). \tag{3.31}$$

where $w(t)$ is the weighting of the squared difference. The Cramér-Von Mises (CVM) statistic is figured with $w(t) = 1$

Consider $F(t_i;\theta)$ to be a known CDF whiles the k-dimensional parameter is unknown. The CVM test statistic is given as;

$$CVM = C^2\left(1 + \frac{1}{2n}\right), \tag{3.32}$$

where

$$C^2 = \sum_{i=1}^{n}\left(\left(Z_i - \frac{2_i - 1}{2n}\right)^2 + \frac{1}{12n}\right)$$

and

$$Z_i = Q^{-1}F^*((x_i, \theta)).$$

This statistics can be obtained by the following steps;

i. Arrange the observations in ascending order and determine $F(t_i; \theta)$

ii. Estimate $Z_i = Q^{-1}(F^*((t_i, \theta)))$ where $Q^{-1}(.)$ is the quantile function

iii. Compute C^2 and CVM

If the CVM statistic is higher than the tabularized value, H_0 is rejected. When comparing tentative distributions, the distribution with the smallest CVM is the best and adequate distribution.

3.8.3 Anderson-Darling Test

When dealing the KS statistic, the theoretical and empirical CDFs are usually flat at the tails of the distribution hence the maximum deviation of the KS test is likely to occur in the tails of the distribution. For a chi-square test, the empirical frequencies must be grouped at the tails. Due to this the KS and chi-square test might not reveal any differences between the empirical and the hypothesised frequencies at the tails of the hypothesised distribution even if differences actually exist. The Anderson-Darling (AD) statistic introduced by Anderson and Darling (1954) solves this problem since it place more weight or discriminating power at the tails of the distribution considered. This test is of great importance if the tails of a specified distribution is of interest. The AD test statistics is also based on the squared integral of the difference between the empirical distribution and the hypothesised distribution. However, the AD test has its weighted function given as the inverse of the odd function, thus;

$$w(t) = [(F^*(t_i))(1 - F^*(t_i))]^{-1}.$$

The AD test involves the following steps;

i. Arrange the data set in ascending order

ii. Determine the CDF of the hypothesised distribution $F^*(t_i), \; i = 1, 2, .., n$

iii. The AD statistics is given by $A = -n - \frac{1}{n}\sum_{i=1}^{n}[(2_i - 1)]lnF^*(t_i) + ln(1 - F^*(t_{n+1-i}))$

iv. Obtain the table value from the considered distribution for a selected significance level.

v. Compare the test statistic with the critical value, reject H_o if the test statistic is greater than the critical value.

3.9 Application Data and Source

Four data set were used in this study to determine the validity of the developed distribution. The NHGPW distribution was applied to two data set; thus 101 observations representing the failure time (in hours) of Kevlar 49/epoxy strands subjected to constant sustained pressure at 90 percent stress level. This data was first presented by Barlow et al. (1984) and Andrews and Herzberg (2012) and also applied by Mdlongwa et al. (2018) and Nasiru (2018). The second data set are failure times data of 84 aircraft windshield. This data was given in Murthy et al. (2004).

The PGPW class of distributions were also applied on two data set. The first application involves 30 observations from aircraft air conditioning system failure times. The second applications used failure data on 63 aircraft service times given in Murthy et al. (2004) and recently studied by Tahir et al. (2015). These data sets are presented in in Appendix A.

CHAPTER 4

The NADARAJAH HAGHIGHI GENERALISED POWER WEIBULL DISTRIBUTION

4.1 Introduction

This chapter presents the NHGPW distribution which compounds the NH and GPW distributions. In this approach, we have a composition by taking the minimum of two continuous independent random variables. The statistical properties, estimators of parameters and application of this distribution to lifetime data are also presented.

4.2 The Nadarajah Haghighi Generalised Power Weibull Distribution

Assume we have a series system with dual components with independently distributed lifetime failure variables T_1 and T_2. Since the components are connected in series and the main system fails if any one or both sub-components fail, the minimum failure time is modelled to check the time of failure of the main system. We assume that T_1 and T_2 are independent random variables. Hence, the stochastic representation of their failure rate distribution is;

$$T = min(T_1, T_2). \tag{4.1}$$

Since the two components must be working for the system success, the main system's reliability function is the product of the individual/marginal reliability of the sub-components (Dimitri and Prentice,1991; Paul and David, 2003; Patrick, 2002). Thus;

$$S(t) = e^{-\int_0^x h_1(t)dt} \times e^{-\int_0^x h_2(t)dt} \tag{4.2}$$

$$S(t) = e^{-\left[\int_0^x (h_1(t)+h_2(t))dt\right]}dt.$$

Assuming for these two components operating in series, the failure rate of component one follows the NH distribution ($T_1 \sim NH(\alpha, \beta)$) and that of component two follows the GPW distribution ($T_2 \sim PGW(\lambda, \theta, \gamma)$) represented as $h_1(t)$ and $h_2(t)$ respectively. These are given as;

$$h_1(t) = \alpha\beta(1 + \alpha t)^{\beta-1}. \tag{4.3}$$

$$h_2(t) = \lambda\gamma\theta t^{\gamma-1}(1 + \lambda t^{\gamma})^{\theta-1}. \tag{4.4}$$

Then a new distribution can be developed to model the joint reliability (survival) function of this system. The reliability/survival function of this new distribution (Nadarajah Haghighi Generalised Power Weibull (NHGPW)) which combines the NH and GPW distributions is given as;

$$s(t) = e^{-\left[\int_0^x (((\alpha\beta(1+\alpha t)^{\beta-1}) + (\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}))dt\right]} \tag{4.5}$$

Equation (4.5) is solved using integration by substitution as shown below. Firstly, we represent the first part of equation (4.5) by $H_1(t)$.

$$H_1(t) = \int_0^x \alpha\beta(1 + \alpha t)^{\beta-1} dt,$$

letting

$$u = 1 + \alpha t, then \left\{ \begin{array}{ll} \text{t} \longrightarrow 0, & \text{u} \longrightarrow 1 \\ \text{t} \longrightarrow x, & \text{u} \longrightarrow 1 + \alpha x \end{array} \right\}$$

,

Also,

$$\begin{aligned} \frac{du}{dt} &= \alpha, \\ \Rightarrow \frac{du}{\alpha} &= dt, \end{aligned}$$

hence, $H_1(t)$ becomes;

$$\begin{aligned} H_1(t) &= \alpha\beta\int_1^{1+\alpha x} u^{\beta-1}\frac{du}{\alpha} \\ &= \beta\left[\frac{u^\beta}{\beta}\right]_1^{1+\alpha x} \\ &= (1+\alpha x)^\beta - 1. \end{aligned}$$

We also represent the second part of equation (4.5) by $H_2(t)$. Thus;

$$H_2(t) = \int_0^x \lambda\theta\gamma t^{\gamma-1}\left(1+\lambda t^\gamma\right)^{\theta-1} dt,$$

also letting

$$y = 1+\lambda t^\gamma, then \left\{ \begin{array}{ll} \text{t} \longrightarrow 0, & \text{y} \longrightarrow 1 \\ \text{t} \longrightarrow x, & \text{y} \longrightarrow 1+\lambda x^\gamma \end{array} \right\}$$

,

in addition,

$$\frac{dy}{dt} = \gamma\lambda t^{\gamma-1},$$

$$\frac{dy}{\gamma\lambda t^{\gamma-1}} = dt,$$

hence,

$$\begin{aligned} H_2(t) &= \int_1^{1+\lambda x} \theta y^{\theta-1} dy \\ &= \theta\left[\frac{y^\theta}{\theta}\right]_1^{1+\lambda x^\gamma} \\ &= (1+\lambda t^\gamma)^\theta - 1. \end{aligned}$$

Inputting the expanded forms of $H_1(t)$ and $H_2(t)$ in equation 4.5, the survival expression of the NHGPW distribution is;

$$s(x) = e^{-\left[(1+\alpha x)^\beta - 1\right] + \left[(1+\lambda x^\gamma)^\theta - 1\right]}, \tag{4.6}$$

$$\Rightarrow s(x) = e^{-\left[(1+\alpha x)^\beta + (1+\lambda x^\gamma)^\theta - 2\right]}. \tag{4.7}$$

The CDF of the new distribution can be obtained from its survival function as;

$$F(x) = 1 - s(x), \tag{4.8}$$

therefore the distribution function (CDF) of the NHGPW distribution is;

$$F(x) = 1 - e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}, x > 0. \tag{4.9}$$

Where $\beta > 0, \gamma > 0, \theta > 0$ shape parameters and $\alpha > 0, \lambda > 0$ are scale parameters.

Lemma 4.1. The NHPGW distribution has a well defined CDF.

Proof. For the CDF of the NHGPW distribution to be well defined, then it must satisfy the basic properties of probability distribution; thus, it should be differentiable, monotonically non-decreasing and should be bounded between 0 and 1 for the support interval of the random variable (which is $x > 0$ for the NHGPW distribution). Thus; $x \longrightarrow \infty, F(x) = 1$ and $x \longrightarrow 0, F(x) = 0$.

As $x \to \infty$, the limits of the CDF of the NHGPW distribution is given as;

$$\begin{aligned}
\lim_{x\to\infty} F(x) &= \lim_{x\to(\infty)} \left(1 - e^{-[(1+\alpha(\infty))^{\beta}+(1+\lambda(\infty)^{\gamma})^{\theta}-2]}\right) \\
&= [1 - e^{-((\infty)-2)}] \\
&= [1 - e^{-(\infty)}] \\
&= 1
\end{aligned}$$

Also as $x \to 0$, the limits of the NHGPW CDF is expressed as;

$$\begin{aligned}
\lim_{x\to 0} F(x) &= \lim_{x\to 0} \left(1 - e^{-[(1+\alpha(0))^{\beta}+(1+\lambda 0^{\gamma})^{\theta}-2]}\right) \\
&= \lim_{x\to(0)} \left(1 - e^{-[(1+\alpha(0))^{\beta}+(1+\lambda(0)^{\gamma})^{\theta}-2]}\right) \\
&= [1 - e^{-[(2)-2])}] \\
&= [1 - e^{-(0)}] \\
&= 0
\end{aligned}$$

Therefore, $F(x)$ is a valid CDF.

By differentiating the CDF of the NHGPW distribution, we obtain its PDF given as;

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) \\ &= \frac{d}{dx}(1 - e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}) \end{aligned}$$

$$\Rightarrow f(x) = \left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \times e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}, x > 0. \quad (4.10)$$

Lemma 4.2. The PDF of the NHPGW distribution is well defined.

Proof. The PDF of the NHPGW distribution satisfies the following;

- $f(x) \geq 0$, thus $f(x)$ is non-negative
- $\int_{-\infty}^{\infty} f(x)dx = 1$, thus the integration over the support values of the random variable is 1.

For $x > 0$,

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} (\left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \times e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]})dx$$

If

$$y = (1+\alpha x)^{\beta} + (1+\lambda x^{\gamma})^{\theta} - 2$$

then;

$$\left\{ \begin{array}{ll} \text{x} \longrightarrow \infty, & y = \infty \\ \text{x} \longrightarrow 0, & y = 0 \end{array} \right\}.$$

Also,

$$\frac{dy}{dx} = \alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}$$

and

$$dx = \frac{dy}{[\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}]}.$$

Therefore $\int_0^\infty f(x)dx$ is;

$$\begin{aligned}
&= \int_0^\infty (\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1})e^{-[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2]} \\
&\cdot \frac{dy}{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}} \\
&= \int_0^\infty e^{-y}dy \\
&= \left[-e^{-y}\right]_0^\infty \\
&= [-0-(-1)] \\
&= 1.
\end{aligned}$$

Therefore, the PDF of the NHGPW distribution is well defined.

Proposition 4.1. The limits of the PDF of the NHGPW distribution for various parameter values are;

$$\lim_{x\to 0} f(x) = \begin{cases} \infty & \gamma < 1 \\ \alpha\beta + \lambda\theta & \gamma = 1 \\ \alpha\beta & \gamma > 1 \end{cases}$$

and if $x \to \infty$, then $f(x) = 0$.

Proof.

Using

$$f(x) = \left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}\right\} e^{-\left[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2\right]}, x > 0$$

$x \to 0$

$$f(x) = \alpha\beta + \lambda\gamma\theta x^{\gamma-1}$$

$\gamma < 1$

$$f(x) = \infty$$

$\gamma = 1$

$$f(x) = \alpha\beta + \lambda\theta$$

$\gamma > 1$

$$f(x) = \alpha\beta$$

But if $x \longrightarrow \infty$, then we have;

$$\begin{aligned} f(x) &= \left\{\alpha\beta(1+\alpha(\infty))^{\beta-1} + \lambda\gamma\theta(\infty)^{\gamma-1}(1+\lambda(\infty)^{\gamma})^{\theta-1}\right\} e^{-\left[(1+\alpha(\infty))^{\beta}+(1+\lambda(\infty)^{\gamma})^{\theta}-2\right]} \\ &= (\infty) + (\infty) \times ((\infty)^{\theta-1}) \times e^{-(\infty)} = (\infty) \times 0 \\ &= 0. \end{aligned}$$

The plot of the PDF of the NHGPW distribution is shown in Figure (4.1) and Figure (4.2). It is evident that, for various parameter values, the PDF of the NHGPW distribution distribution can be symmetric, positively skewed, bathtub, unimodal or modified bathtub. The PDF also showed various skewness and kurtosis based on different combination of parameter values.

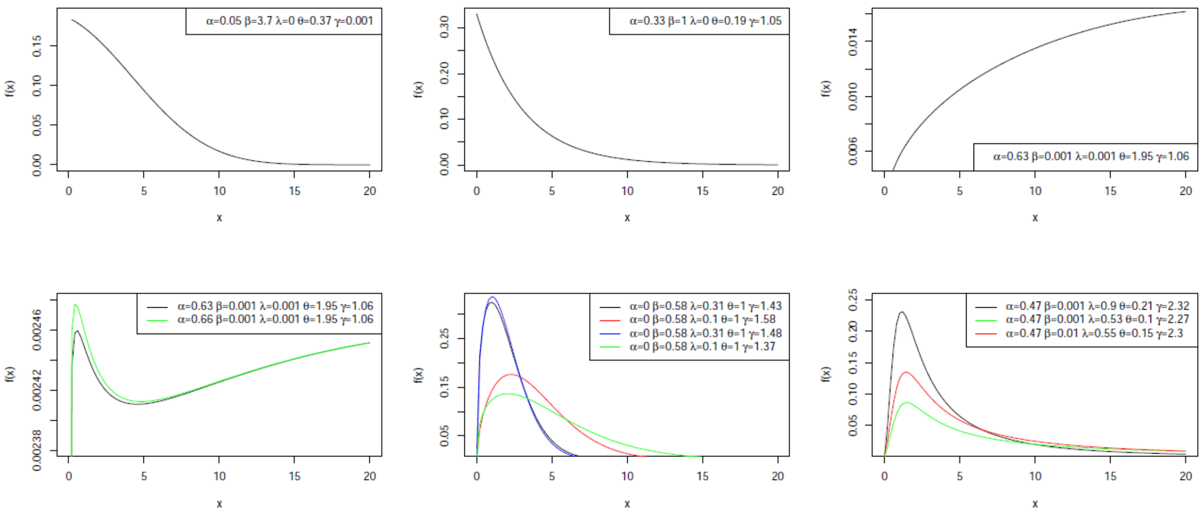

Figure 4.1: **PDF plots of the NHGPW distribution**

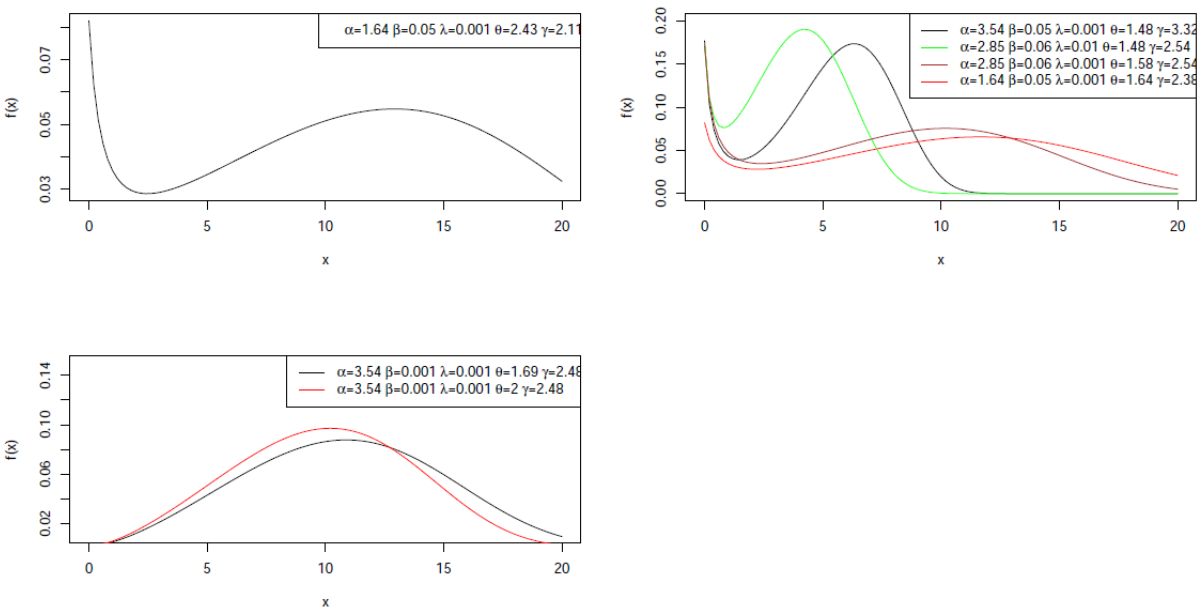

Figure 4.2: **PDF plots of the NHGPW distribution**

The hazard rate function of the NHGPW distribution is given as:

$$\begin{aligned} h(x) &= \frac{f(x)}{s(x)} \\ &= \frac{\left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \times e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}}{e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}} \end{aligned}$$

$$h(x) = \alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}. \tag{4.11}$$

The hazard function gives the instantaneous failure rate per unit time. It measures the event rate at time x, conditional on survival until time x $(X \geq x)$. It is often look as the frequency with which a system/component fails per unit time. In practice, it reported as the mean distance between failure times.

The PDF of the NHGPW distribution is therefore related to its hazard rate function as;

$$f(x) = \left\{(h_{NH}(x) + h_{GPW}(x))\, e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}\right\}, x > 0. \tag{4.12}$$

Proposition 4.2. The limits of the hazard function of the NHPGW distribution is given as;

$$\lim_{x \to 0} h(x) = \alpha\beta$$

and

$$\lim_{x \to \infty} h(x) = \begin{cases} 0\,, \beta < 1,\; \gamma < 1, \theta < 1 \\ \infty,\; \beta > 1, \gamma > 1,\; \theta > 1 \\ \infty,\; \beta > 1, \gamma = 1,\; \theta > 1 \\ 0,\; \beta < 1, \gamma = 1,\; \theta < 1 \\ \alpha\beta + \lambda\gamma\theta,\; \beta = 1, \gamma = 1,\; \theta = 1 \end{cases}.$$

Figure (4.3) to figure (4.4) shows different shapes of the hazard function of the NHGPW distribution. It is seen that, for different parameter value combination of its parameter values, the hazard function can be constant, monotonically increasing, monotonically decreasing, bathtub, unimodal (upside down bathtub) and modified bathtub (bathtub followed by unimodal). This indicates that, the developed NHGPW distribution can adequately model both monotonic and non-monotonic failure rates which are often encountered in lifetime data related to system connected in series.

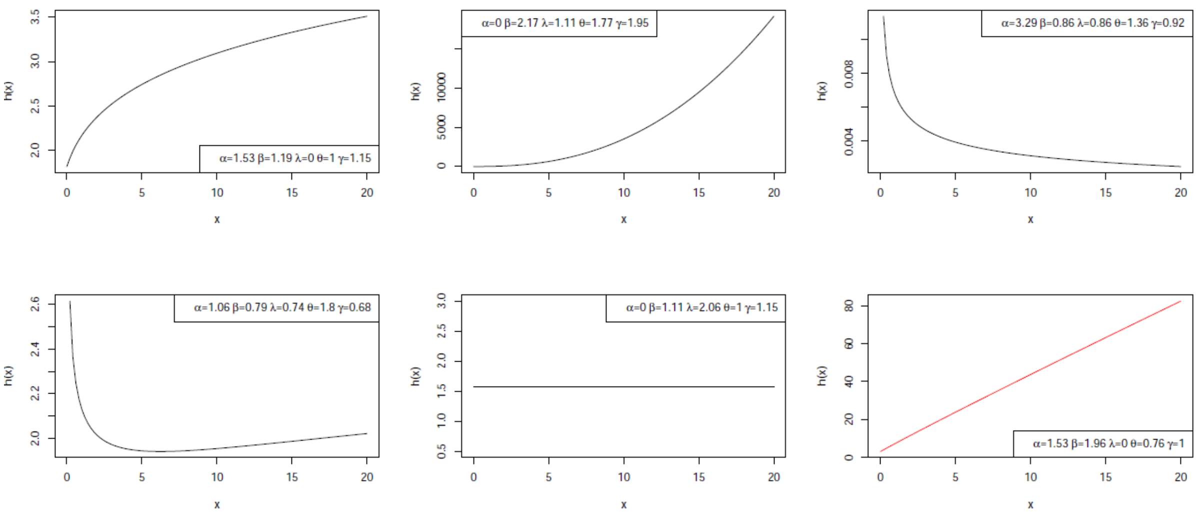

Figure 4.3: **Hazard function of the NHGPW distribution**

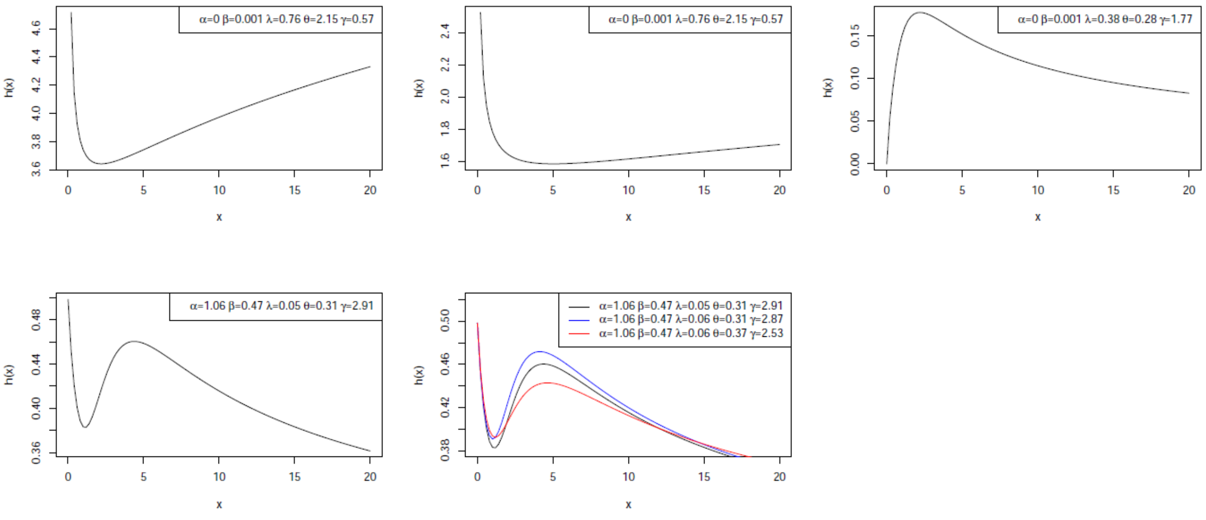

Figure 4.4: **Hazard function of the NHGPW distribution**

4.2.1 Sub-distributions of the NHGPW Distribution

The NHGPW distribution has as sub-distributions a number of existing and new lifetime distributions for modeling lifetime dataset. Some of these distributions are;

1. the generalised power Weibull distribution.

   The NHGPW distribution reduces to the GPW distribution if either $\beta = 0$ or $\alpha = 0$ with the CDF of the GPW distribution given as;

$$F(x) = 1 - e^{[1-(1+\lambda x^{\gamma})^{\theta}]}, x, \lambda, \gamma, \theta > 0 \tag{4.13}$$

2. the Nadarajah Haghighi Distribution.

   If $\lambda = 0$, or $\theta = 0$ the NHGPW distribution reduces to the NH distribution with CDF given as;

$$F(x) = 1 - e^{[1-(1+\alpha x)^{\beta}]}, x, \alpha, \beta > 0 \tag{4.14}$$

3. the Exponential distribution

   For $\beta = 1$ and $\lambda = 0$ (or $\theta = 0$), the NHGPW distribution reduces to an exponential distribution with CDF defined as;

$$F(x) = 1 - e^{-\alpha x}, \alpha > 0 \tag{4.15}$$

or

For $\alpha = 0$ (or $\beta = 0$), $\gamma = 1$, and $\theta = 1$, the NHGPW distribution reduces to an exponential distribution with CDF defined as;

$$F(x) = 1 - e^{-\lambda x}, \lambda > 0 \tag{4.16}$$

4. the Weibull distribution.

   For $\alpha = 0$ and $\theta = 1$, the NHGPW distribution reduces to a two parameter Weibull distribution with its CDF given as;

$$F(x) = 1 - e^{-\lambda x^{\gamma}}, x, \lambda, \gamma > 0 \tag{4.17}$$

5. the Linear failure rate distribution.

   For $\beta = 1, \gamma = 2$ and $\theta = 1$, the NHGPW distribution reduces to the linear failure rate distribution with CDF defined as;

$$F(x) = 1 - e^{-\left[\alpha x + \lambda x^2\right]}, x, \alpha, \lambda > 0 \tag{4.18}$$

6. the Rayhigh distribution.

   For $\beta = 1, \alpha = 0, \gamma = 2$, and $\theta = 1$, the NHGPW distribution is equivalent to the Rayhigh distribution with CDF defined as;

$$F(x) = 1 - e^{\lambda x^2}, x, \lambda > 0 \tag{4.19}$$

7. The Generalized power Rayhigh distribution.

   For $\alpha = 0$ and $\gamma = 2$, the NHGPW distribution equates to the Generalized power Rayhigh distribution with the following CDF;

$$F(x) = 1 - e^{\left[1-(1+\lambda x^2)^{\theta}\right]}, x, \lambda > 0, \gamma > 0 \tag{4.20}$$

8. The NH-NH distribution (New distribution)

   For, $\gamma = 1$, a new distribution called the NH-NH distribution can be obtained from the NHGPW distribution with its CDF given as;

$$F(x) = 1 - e^{-[1-(1+\gamma x)^{\beta}+(1+\lambda x)^{\theta}-2]}, x, \gamma >, \beta, \lambda, \theta > 0 \tag{4.21}$$

   This distribution compounds two NH distributions and is applicable for modelling failure rates in a system with two components with failure rate following the NH distributions

9. The Exponential-Exponential Distribution (New distribution)

   For $\beta = 1, \gamma = 1$ and $\theta = 1$, another new distribution termed the exponential exponential distribution can be derived from the NHGPW distribution with CDF defined as;

$$F(x) = 1 - e^{-[\alpha x+\lambda x]}, x, \alpha, \lambda > 0 \tag{4.22}$$

Table 4.1: **Sub-Distributions of the NHGPW Distribution**

Model	α	β	λ	γ	θ
Generalized power Weibul	0	β	λ	γ	θ
Nadarajah-Haghighi	α	β	0	γ	θ
NH-NH	α	β	λ	1	θ
Exponential	α	1	0	γ	θ
Exponential	0	β	λ	1	1
Weibull	0	β	λ	γ	1
Exponential-Exponential	α	1	λ	1	1
Linear Failure Rate	α	1	λ	2	1
Rayhigh Distribution	0	1	λ	2	1
Generalized Power Rayhigh	0	β	λ	2	θ

4.3 Statistical Properties of the NHGPW Distribution

Various statistical properties such as the quantile function, moments, Moment generating function and order statistics for the NHGPW distribution were derived.

4.3.1 Quantiles of the NHGPW distribution

The quantile function is the inverse of the cumulative distribution function. The quantile function can be used in both statistical applications and Monte carlo methods. It can be used for generating random numbers from a given distribution. It can as well serve as an alternative way of describing a probability distribution other than the probability density function and cumulative distribution function or characteristic function. In situations where the moment of a random variable does not exist, it can be used to compute the measures of skewness and kurtosis. Quantiles can also be used to determine the quartiles (lower quartile, median, upper quartile), interquartile range and percentiles. Quantiles have advantages compared to the classical measures of skewness and kurtosis since they are not much affected by outliers and they always exist for a distribution even lacking defined moments.

Proposition 4.3. The Quantile function of the NHGPW distribution for $p\epsilon[0,1]$is obtained by solving the equation below.

$$(1+\alpha x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta}+log(1-p)-2=0, p\epsilon[0,1]. \tag{4.23}$$

Proof. Using the CDF of the NHGPW distribution and considering $F(x_p)$ and $p\epsilon[0,1]$, the quantile function of the NHGPW distribution can be expressed as;

$$1-e^{-\left[(1+\alpha x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta}-2\right]}=p.$$

We make the exponent of the expression the subject by using the steps below;

$$\begin{aligned} e^{-\left[(1+\alpha x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta}-2\right]} &= 1-p \\ -\left[(1+\alpha x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta}-2\right] &= \log(1-p) \\ (1+\alpha x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta}-2 &= -\log(1-p) \\ (1+\alpha x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta} &= 2-\log(1-p), \end{aligned}$$

which can also be written as;

$$(1+\gamma x_p)^{\beta}+(1+\lambda x_p^{\gamma})^{\theta}+log(1-p)-2=0, p\epsilon[0,1].$$

Since the quantile function does not have a closed form, we solve it using numerical method (Newton Raphson estimation approach). With this approach, some random numbers were generated using the quantile function in equation (4.23) for five different parameter value combinations of the NHGPW distribution. The parameter value combinations were then used to obtain the quantiles.The Booleys skewness (B.Sk) and Moors kurtosis (M.Ku) values for the different parameter values were also calculated.

The Bowleys skewness and Moors' kurtosis measured based on the quantile function are given respectively as;

$$B.Sk=\frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}. \tag{4.24}$$

$$M.Ku=\frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{3}{4})-Q(\frac{1}{4})}. \tag{4.25}$$

The results are presented in Table 4.2.

Table 4.2: **NHGPW Quantiles for Selected Parameter Values** $(\alpha, \beta, \lambda, \theta, \gamma)$

p	(2.5, 10.3,4.5, 0.1,0.5)	(3.54, 5.3, 6.5, 2.1,8.5)	(2.0, 2.0, 3.5, 0.5,2.0)	(4.0, 3.0, 5.0, 4.0,0.8)	(3.54, 0.05,0.01, 1.46,3.32)
0.1	0.003	0.005	0.025	0.001	1.126
0.2	0.007	0.011	0.052	0.003	1.860
0.3	0.0108	0.017	0.080	0.005	2.299
0.4	0.015	0.023	0.110	0.007	2.640
0.5	0.020	0.030	0.144	0.010	2.939
0.6	0.025	0.037	0.182	0.013	3.220
0.7	0.031	0.045	0.228	0.017	3.505
0.8	0.038	0.056	0.287	0.022	3.820
0.9	0.048	0.071	0.377	0.030	4.225
B.Sk	0.136	-6.106	1.383	0.239	-0.079
M.Ku	1.144	1.160	1.203	1.230	1.291

4.3.2 Mixture representation of the NHGPW probability density function

In order to obtain other statistical properties of the NHGPW distribution, further expansion of its PDF into a simple form is necessary. The NHGPW PDF given as;

$$\Rightarrow f(x) = \left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \times e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}, x > 0.$$

can also be written as;

$$f(x) = f_1(x) + f_2(x), \tag{4.26}$$

where,

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1}e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]},$$

$$= \alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}e^{1-(1+\lambda x^{\gamma})^{\theta}}. \tag{4.27}$$

and,

$$f_2(x) = \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}. \tag{4.28}$$

Lemma 4.3. Using Taylor series and the generalized form of binomial expansion, $f_1(x)$ and $f_2(x)$ can be expanded into;

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} x^{\gamma\theta(i-j)-\gamma k} w_{ijk} e^{1-(1+\alpha x)^{\beta}}. \tag{4.29}$$

and

$$f_2(x) = \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} x^{\beta(i-j)-k}.w^{*}_{ijk} e^{1-(1+\lambda x^{\gamma})^{\theta}}. \tag{4.30}$$

where $w_{ijk} = \frac{(-1)^{i+j}\lambda^{\theta(i-j)-k}}{i!}\binom{i}{j}\binom{\theta(i-j)}{k}$ and $w^{*}_{ijk} = \frac{(-1)^{i+j}\alpha^{\beta(i-j)-k}}{i!}\binom{i}{j}\binom{\beta(i-j)}{k}$.

Proof; Using

$$f(x) = \left\{\alpha\beta(1+\alpha x)^{\beta-1} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \times e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}, x > 0$$

$f(x)$ can also be written as;

$$f(x) = \alpha\beta(1+\alpha x)^{\beta-1}e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]} + \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}, x > 0.$$

Thus the PDF is divided into;

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1}e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}$$

$$= \alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}e^{1-(1+\lambda x^{\gamma})^{\theta}}$$

But the Taylor series expansion of an exponential function, e^{x} is; $\sum_{i=0}^{\infty}\frac{x^{i}}{i!}$. Hence the Taylor series expansion of $f_1(x)$ is;

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}.\sum_{i=0}^{\infty}\frac{[1-(1+\lambda x^{\gamma})^{\theta}]^{i}}{i!}$$

which can also be written as;

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \sum_{i=0}^{\infty} \frac{(-1)^i[(1+\lambda x^\gamma)^\theta - 1]^i}{i!}$$

using the generalised form of binomial expansion;

$$(x+y)^i = \sum_{j=0}^{i} \binom{i}{j} x^{i-j} y^j, (|x| > |y|) \tag{4.31}$$

where $y = (-1)$ and $x = (1+\lambda x^\gamma)^\theta$ to further expand $f_{1a}(x)$ we get; we get;

$$\begin{aligned} f_1(x) &= \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{j=0}^{i} \binom{i}{j} \left[(1+\lambda x^\lambda)^\theta\right]^{i-j} (-1)^j \\ &= \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{i!} \binom{i}{j} (\lambda x^\gamma + 1)^{\theta(i-j)} \end{aligned}$$

Using the generalised form of the binomial expansion given in equation (4.31) where $y = 1$ and $x = \lambda x^\gamma$, we further expand $f_1(x)$ into;

$$\begin{aligned} f_1(x) &= \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{i!} \binom{i}{j} \sum_{k=0}^{\infty} \binom{\theta(i-j)}{k} (\lambda x^\gamma)^{\theta(i-j)-k} (1)^k \\ &= \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{i}{j} \binom{\theta(i-j)}{k} \lambda^{\theta(i+j)-k} x^{\gamma[\theta(i-j)-k]} \\ &= \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^\beta} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{i+j} \lambda^{\theta(i-j)-k} x^{\gamma[(\theta(i-j)-k)]}}{i!} \binom{i}{j} \binom{\theta(i-j)}{k}. \end{aligned}$$

which can also be written as;

$$f_1(x) = \alpha\beta(1+\alpha x)^{\beta-1} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} x^{\gamma\theta(i-j)-\gamma k} w_{ijk} e^{1-(1+\alpha x)^\beta}.$$

where $w_{ijk} = \frac{(-1)^{i+j}\lambda^{\theta(i-j)-k}}{i!}\binom{i}{j}\binom{\theta(i-j)}{k}$

Using the same approach to expand $f_2(x)$, we obtain;

$$\begin{aligned}
f_2(x) &= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{-[(1+\alpha x)^\beta+(1+\lambda x^\gamma)^\theta-2]} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{1-(1+\alpha x)^\beta}e^{1-(1+\lambda x^\gamma)^\theta} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{1-(1+\lambda x^\gamma)^\theta}\sum_{i=0}^{\infty}\frac{(-1)^i[(1+\alpha x)^\beta-1]^i}{i!} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{1-(1+\lambda x^\gamma)^\theta}\sum_{i=0}^{\infty}\frac{(-1)^{i+j}}{i!}\sum_{j=0}^{i}\binom{i}{j}(1+\alpha x)^{\beta(i-j)} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{1-(1+\lambda x^\gamma)^\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{i!}\binom{i}{j}(\alpha x+1)^{\beta(i-j)} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{1-(1+\lambda x^\gamma)^\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{i!}\binom{i}{j}\sum_{k=0}^{\infty}\binom{\beta(i-j)}{k}(\alpha x)^{\beta(i-j)k} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}e^{1-(1+\lambda x^\gamma)^\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}}{i!}\binom{i}{j}\binom{\beta(i-j)}{k}\alpha^{\beta(i-j)-k}x^{\beta(i-j)-k} \\
&= \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}\alpha^{\beta(i-j)-k}}{i!}\binom{i}{j}\binom{\beta(i-j)}{k}x^{\beta(i-j)-k}e^{1-(1+\lambda x^\gamma)^\theta}.
\end{aligned}$$

Which can also be expressed as;

$$f_2(x) = \lambda\gamma\theta x^{\gamma-1}(1+\lambda x^\gamma)^{\theta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}x^{\beta(i-j)-k}.w^*_{ijk}e^{1-(1+\lambda x^\gamma)^\theta},$$

where $w^*_{ijk} = \frac{(-1)^{i+j}\alpha^{\beta(i-j)-k}}{i!}\binom{i}{j}\binom{\beta(i-j)}{k}$.

4.3.3 Moments of the NHGPW distribution

Moments of a random variable are very important in statistical analysis. They can be used in measuring central tendency/location, variation, skewness, kurtosis and other statistical procedures. The moment of a random variable X is the expectations of the r^{th} values of the random variable if the expectation exists $(r = 1, 2, ...,)$. The moment might not exist if the distribution is heavy tailed and highly skewed. The moment can be about the origin or about the mean. For a continuous random variable X with associated PDF, $f(x)$, the

r^{th} non-central moment (ordinary moment) of the distribution of X, is given by;

$$\mu_r^{'} = \int_0^{\infty} x^r f(x)dx. \tag{4.32}$$

Proposition 4.4. The r^{th} non-central moment of NHGPW distribution is given as;

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[w_{ijk}A\Gamma\left(\frac{m}{\beta}+1,1\right)+w_{ijk}^{*}B\Gamma\left(\frac{m}{\theta}+1,1\right)\right]. \tag{4.33}$$

where

$A = e\alpha^{-r-r\theta(i-j)+\gamma k}(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m}$ and

$B = e\lambda^{\frac{-\gamma-\beta(i-j)+k}{\gamma}}(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}$.

Proof. The r^{th} non-central moment of a continuous random variable is given as;

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx.$$

For the NHGPW distribution, we have the moment expressed as;

$$\mu_r^{'} = \int_0^{\infty} x^r f_1(x)dx + \int_0^{\infty} x^r f_2(x)dx.$$

Using the expanded form of f_1x defined in equation (4.29), we have;

$$\begin{aligned}\int_0^{\infty} x^r f_1(x)dx &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk}\int_0^{\infty}\alpha\beta(1+\alpha x)^{\beta-1}x^{r+\gamma\theta(i-j)-\gamma k}e^{1-(1+\alpha x)^{\beta}}dx\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk}\alpha\beta e\int_0^{\infty}(1+\alpha x)^{\beta-1}x^{r+\gamma\theta(i-j)-\gamma k}e^{-(1+\alpha x)^{\beta}}dx.\end{aligned}$$

We use integration by substitution to solve the equation as shown below;

Let

$$y = (1+\alpha x)^{\beta}, x = \frac{1}{\alpha}\left(y^{\frac{1}{\beta}}-1\right)$$

also,

$$\left\{\begin{array}{ll} \text{x}\longrightarrow 0, & y\rightarrow 1\\ \text{x}\longrightarrow \infty, & y\rightarrow\infty\end{array}\right\}$$

and

$$\begin{aligned} \frac{dy}{dx} &= \alpha\beta(1+\alpha x)^{\beta-1} \\ \Rightarrow dx &= \frac{dy}{\alpha\beta(1+\alpha x)^{\beta-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty x^r f_1(x)dx &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk} e \int_0^\infty \alpha\beta(1+\alpha x)^{\beta-1}\left[\frac{1}{\alpha}\left(y^{\frac{1}{\beta}}-1\right)\right]^{r+\gamma\theta(i-j)-\gamma k} e^{-y}\frac{dy}{\alpha\beta(1+\alpha x)^{\beta-1}} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk} e \int_1^\infty \left[\frac{1}{\alpha}\left(y^{\frac{1}{\beta}}-1\right)\right]^{r+\gamma\theta(i-j)-\gamma k} e^{-y}dy \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk} e\alpha^{-r-\gamma\theta(i-j)+\gamma k} \int_1^\infty \left(y^{\frac{1}{\beta}}-1\right)^{r-\gamma\theta(i-j)-\gamma k} e^{-y}dy \end{aligned}$$

using the generalised form of binomial expansion, $(x+y)^i = \sum_{j=o}^{i}\binom{i}{j} y^{i-j}x^j, (\mid y\mid>\mid x\mid)$. With $y=(-1)$ and $x = y^{\frac{1}{\beta}}$ to expand the expression above we have;

$$\begin{aligned} &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w_{ijk} e\alpha^{-r-\gamma\theta(i-j)+\gamma k} \int_1^\infty \sum_{m=0}^{\infty}(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m} y^{\frac{m}{\beta}}e^{-y}dy \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w_{ijk} e\alpha^{-r-\gamma\theta(i-j)+\gamma k}(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m} \\ &\times \int_1^\infty y^{(\frac{m}{\beta}+1)-1}e^{-y}dy \end{aligned}$$

But $\int_1^\infty y^{[(\frac{m}{\beta}+1)-1]}e^{-y}dy = \int_1^\infty y^{b-1}e^{-y}dy$ is a complementary gamma function given as; $\Gamma(b,a)$, where b and a are the parameters. For the $f_1(x)$ expression above, $b = \frac{m}{\beta}+1$. whiles $a=1$.

Therefore,

$$\begin{aligned} \int_0^\infty x^r f_1(x)dx &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty} w_{ijk} e\alpha^{-r-\gamma\theta(i-j)+\gamma k}(-1)^{r+\gamma\theta(i-j)-\gamma k-m}\binom{r+\gamma\theta(i-j)-\gamma k}{m} \\ &\times \Gamma\left(\frac{m}{\beta}+1,1\right). \end{aligned}$$

Also using the expanded form of $f_2(x)$ defined in equation (4.30), we have;

$$\int_0^{\infty} x^r f_2(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w^*_{ijk}\lambda\gamma\theta\int_0^{\infty} x^{r+\gamma-1+\beta(i-j)-k}(1+\lambda x^{\gamma})^{\theta-1}e.e^{-(1+\lambda x^{\gamma})^{\theta}}dx$$

Using integration by substitution to solve, $f_2(x)$, we take the following steps.

Let

$$y=(1+\lambda x^{\gamma})^{\theta}, x=\lambda^{-\frac{1}{\gamma}}(y^{\frac{1}{\theta}}-1)^{\frac{1}{\gamma}}$$

and

$$\left\{\begin{array}{ll} \text{x}\longrightarrow 0, & y\rightarrow 1 \\ \text{x}\longrightarrow \infty, & y\rightarrow\infty \end{array}\right\}$$

Also,

$$\begin{aligned}\frac{dy}{dx} &= \theta\lambda\gamma x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\\ \Rightarrow dx &= \frac{dy}{\theta\lambda\gamma x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{\infty} x^r f_2(x)dx &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w^*_{ijk}e\int_1^{\infty}\left[\lambda^{-\frac{1}{\gamma}}(y^{\frac{1}{\theta}}-1)^{\frac{1}{\gamma}}\right]^{r+\beta(i-j)-k}e^{-y}dy\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w^*_{ijk}e\lambda^{\frac{-r-\beta(i-j)+k}{\gamma}}\int_1^{\infty}\left(y^{\frac{1}{\theta}}-1\right)^{\frac{r+\beta(i-j)-k}{\gamma}}e^{-y}dy\end{aligned}$$

Using the generalised form of binomial expansion; $(x+y)^i=\sum_{j=o}^{i}\binom{i}{j}y^{i-j}x^j, (\mid y\mid>\mid x\mid)$ to expand the expression above we have;

$$\begin{aligned}\int_0^{\infty} x^r f_2(x)dx &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} w^*_{ijk}e\lambda^{\frac{-r-\beta(i-j)+k}{\gamma}}\int_1^{\infty}\sum_{m=0}^{\infty}(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}y^{\frac{m}{\theta}}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}e^{-y}dy\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w^*_{ijk}e\lambda^{\frac{-r-\beta(i-j)+k}{\gamma}}(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}\\ &\times\int_1^{\infty} y^{((\frac{m}{\theta}+1)-1)}e^{-y}dy,\end{aligned}$$

But $\int_1^{\infty} y^{[(\frac{m}{\theta}+1)-1]}e^{-y}dy=\int_1^{\infty} y^{b-1}e^{-y}dy$ is a complementary gamma function given as;

$\Gamma(b, a)$, where b and a are the parameters. For the $f_2(x)$ expression above, $b = \frac{m}{\theta} + 1$. whiles $a = 1$.

Therefore,

$$f_2(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w^*_{ijk} e\lambda^{\frac{-r-\beta(i-j)+k}{\gamma}}(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}\Gamma\left(\frac{m}{\theta}+1, 1\right)$$

$$\mu_r^{'} = \int_0^{\infty} x^r f_1(x)dx + \int_0^{\infty} x^r f_2(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[w_{ijk}A\Gamma\left(\frac{m}{\beta}+1, 1\right) + w^*_{ijk}B\Gamma\left(\frac{m}{\theta}+1, 1\right)\right.$$

Where $r = 1, 2, ...,$ $A = e\alpha^{-r-r\theta(i-j)+\gamma k}(-1)^{r+\gamma\theta}(i-j) - \gamma k - m\binom{r+\gamma\theta(i-j)-\gamma k}{m}$ and $B = e\lambda^{\frac{-\gamma-\beta(i-j)+k}{\gamma}}(-1)^{\frac{r+\beta(i-j)-k}{\gamma}-m}\binom{\frac{r+\beta(i-j)-k}{\gamma}}{m}$

The first five non-central moments obtained by numerical integration of the NHGPW distribution for selected parameter values are presented in Table 4.3. The standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and kurtosis (CK) calculated using these non-central moments are also presented in Table 4.3.

Table 4.3: **Moments of the NHGPW Distribution for Different Parameter Values**

p	(0.1, 0.8,0.8, 0.5,2.4)	(3.54, 0.05, 0.01, 1.4,3.32)	(3.54, 5.3, 6.5, 2.1,8.5)	(2, 2, 3.5, 0.5,2)	(4, 3,5, 4,0.8)
$\mu_1^{'}$	1.476	2.813	0.035	0.178	1.307
$\mu_2^{'}$	3.002	9.280	0.002	0.052	3.217
$\mu_3^{'}$	7.724	33.130	0.001	0.020	1.085
$\mu_4^{'}$	24.049	125.117	1×10^{-05}	0.009	4.502
$\mu_5^{'}$	87.858	494.022	8.9×10^{-07}	0.005	2.190
SD	0.907	1.170	0.026	0.143	1.229
CV	0.614	0.416	0.760	0.807	0.940
CS	1.160	-0.422	0.750	1.203	-3.810
CK	5.086	2.737	3.502	3.813	8.145

The SD, CV, CS and CK are defined respectively as;

$$SD = \sqrt{\mu_2^{'} - (\mu_1^{'})^2}.$$

$$CV = \frac{\sqrt{\mu_2^{'} - (\mu_1^{'})^2}}{\mu_1^{'}}.$$

$$CS = \frac{E[(x-\mu_1^{'})^3]}{[E[(x-\mu_1^{'})^2]^{\frac{3}{2}}} = \frac{\mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2(\mu_1^{'})^3}{((\mu_2^{'} - \mu_1^{'})^2)^{\frac{3}{2}}}.$$

$$CK = \frac{E[(x-\mu_1^{'})^4]}{[E[(x-\mu_1^{'})^2]^2} = \frac{\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6(\mu_1^{'})^2\mu_2^{'} - 3(\mu_1^{'})^4}{((\mu_2^{'} - \mu_1^{'})^2)^2}.$$

4.3.4 Moment Generating function of the NHGPW distribution

The Moment Generating function (MGF) is the function which is used to generate the moments of a distribution. If the moment generating function exist, it is given as;

$$M_x(t) = E(e^{tx}). \tag{4.34}$$

For a continuous distribution, the MGF is expressed as;

$$M_x(t) \;=\; E\left[e^{tx}\right] = \int_0^\infty e^{tx} f(x)dx. \tag{4.35}$$

Proposition 4.5.The moment generating function of the NHGPW distribution is given;

$$M_x(t) \;=\; \sum_{r=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{t^r}{\gamma!}\left[w_{ijk}A\Gamma\left(\frac{m}{\beta}+1,1\right) + w_{ijk}^{*}B\Gamma\left(\frac{m}{\theta}+1,1\right)\right] \tag{4.36}$$

Proof. By definition, the MGF is given as;

$$M_X(t) \;=\; E\left[e^{tx}\right] = \int_0^\infty e^{tx} f(x)dx.$$

Using Taylor series to expand $\int_0^\infty e^{tx} f(x)dx$, we get;

$$\begin{aligned} M_X(t) &= \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r X^r}{r!} f(x)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r f(x)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'} \end{aligned}$$

For the NHGPW distribution,

$$\mu_r^{'} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left[w_{ijk} A\Gamma\left(\frac{m}{\beta}+1,1\right) + w_{ijk}^{*} B\Gamma\left(\frac{m}{\theta}+1,1\right) \right]$$

Hence,

$$M_x(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[w_{ijk} A\Gamma\left(\frac{m}{\beta}+1,1\right) + w_{ijk}^{*} B\Gamma\left(\frac{m}{\theta}+1,1\right) \right]$$

The moment generating function of a random variable X, exist only if the infinite series (in the discrete case) or the improper integral (in the continuous case) is convergent (thus the infinite sequence of the partial sums of the series does have a finite limit). The r^{th} moment about the origin is obtained from the moment generating function by taking the rth derivative of $M_x(t)$ with respect to t and evaluating it at $t = 0$. Thus;

$$E(X^r) = \frac{d^r M_x(t=0)}{dt^r} \qquad (4.37)$$

4.3.5 Order Statistics of the NHGPW Distribution

Order statistics are used to identify the maximum and minimum values (extreme values) of a random variable. They are mostly used in extreme value theory. Let X_1 and X_2 denote the smallest and second smallest value of $(X_1, X_2, ..., X_n)$ and X_p denote the pth smallest value of $(X_1, X_2, ..., X_n)$, then the random variables $X_1, X_2, ..., X_n$, are called the order statistics of the sample $(X_1, X_2, ..., X_n)$ and has pdf of the pth order, given as;

$$f_{p:n}(x) = \frac{n!}{(n-p)!(p-1)!} \left[F(x)\right]^{p-1} \left[1 - F(x)\right]^{n-p} f(x). \qquad (4.38)$$

Proposition 4.6. If $X_1, X_2, ..., X_n$ are random samples from the NHGPW distribution, then the p^{th} order statistic is given as;

$$f_{p;n}(x) = \frac{n!\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}}{(p-1)!(n-p)!}\binom{n-p}{i}\binom{p+i-1}{j}\sum_{i=1}^{n-p}\sum_{j=1}^{p+i-1}(-1)^{1+j}$$

$$\times e^{-(j+1)\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}. \quad (4.39)$$

Proof. By definition, the p^{th} order statistic is given as;

$$f_{p:n}(x) = \frac{n!}{(n-p)!(p-1)!}\left[F(x)\right]^{p-1}\left[1-F(x)\right]^{n-p}f(x).$$

Using the binomial series expansion of $[1-F(x)]^{n-p}$ given as;

$$[1-F(x)]^{n-p} = \sum_{i=1}^{n-p}(-1)^{i}\binom{n-p}{i}[F(x)]^{i}$$

and the condition that, $0 < 1 - F(x) < 1$, we simplify $f_{p;n}(x)$ into;

$$\begin{aligned} f_{p:n}(x) &= \frac{n!}{(n-p)!(p-1)!}\sum_{i=1}^{n-p}(-1)^{i}\binom{n-p}{i}[F(x)]^{i}[F(x)]^{p-1}f(x) \\ &= \frac{n!}{(n-p)!(p-1)!}\sum_{i=1}^{n-p}(-1)^{i}\binom{n-p}{i}[F(x)]^{p+i-1}f(x). \end{aligned}$$

Inputting the CDF and PDF of the NHGPW distribution into $f_{p;n}(x)$, we get;

$$\begin{aligned} f_{p:n}(x) &= \frac{n!}{(n-p)!(p-1)!}\sum_{i=1}^{n-p}(-1)^{i}\binom{n-p}{i}\left[1-e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}\right]^{p+i-1} \\ &\times \left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}, \end{aligned}$$

but $\left[1-e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}\right]^{p+i-1}$ can also be expressed in binomial series expansion form as; $\left[1-e^{-[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2]}\right]^{p+i-1} = \sum_{j=1}^{p+i-1}(-1)^{j}\binom{p+i-1}{j}e^{-j\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}$ inputting this into $f_{p;n}(x)$, we have;

$$\begin{aligned} f_{p;n}(x) &= \frac{n!\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}}{(p-1)!(n-p)!}\sum_{i=1}^{n-p}(-1)^{i}\binom{n-p}{i}\sum_{j=1}^{p+i-1}(-1)^{j}\binom{p+i-1}{j} \\ &\times e^{-j\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}e^{-[(1+\alpha x)^{\beta}+(1+\lambda+x^{\gamma})^{\theta}-2]} \end{aligned}$$

$$= \frac{n!\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}}{(p-1)!(n-p)!}\binom{n-p}{i}\binom{p+i-1}{j}\sum_{i=1}^{n-p}\sum_{j=1}^{p+i-1}(-1)^{1+j}$$
$$\times \quad e^{-(j+1)\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}.$$

Proposition 4.7. The PDF of first order statistics ($p = 1$) is given as;

$$f_{x(1)}(x) = n\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}e^{-n\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}. \qquad (4.40)$$

Proof. The PDF of the first order statistic is defined as;

$$\begin{aligned} f_{x(1)}(x) &= \frac{n!}{(n-1)!(1-1)!}[F(x)]^{1-1}[1-F(x)]^{n-1}f(x) \\ &= n[1-F(x)]^{n-1}f(x) \end{aligned}$$

Imputing the CDF and PDF of the NHGPW distribution into $f_{x(1)}(x)$, we obtain the PDF of the first order statistics as;

$$\begin{aligned} f_{x(1)}(x) &= n\left[e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}\right]^{n-1}\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \\ &\times \quad e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \\ &= n\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}e^{-(n-1)\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \\ &\times \quad e^{-1\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \\ &= n\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}e^{-n\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \end{aligned}$$

Proposition 4.8. The PDF of the largest Order Statistics for the NHGPW distribution ($p = n$) is given as;

$$f_{x(n)}(x) = n\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}\sum_{i=1}^{n-1}(-1)^{i}\binom{n-1}{i}e^{-(i+1)\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}. \qquad (4.41)$$

Proof. The PDF of largest order statistics, ($p = n$) is also expressed as;

$$\begin{aligned} f_{x(n)}(x) &= \frac{n!}{(n-1)!(n-n)!}[F(x)]^{n-1}[1-F(x)]^{n-n}f(x), \\ &= n[F(x)]^{n-1}f(x). \end{aligned}$$

Inputting the CDF and PDF of the NHGPW distribution into $f_{x(n)}(x)$ yields;

$$\begin{aligned} f_{x(n)}(x) &= n\left[1-e^{-[(1+\alpha x)^{\beta}+(1+\lambda+x^{\gamma})^{\theta}-2]}\right]^{n-1}\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\} \\ &\times\ e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \end{aligned}$$

since $[1-e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}]^{n-1}=\sum_{i=1}^{n-1}(-1)^{i}\binom{n-1}{i}[e^{-i\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]}]$, $f_{x(n)}(x)$ becomes;

$$\begin{aligned} f_{x(n)}(x) &= n\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}\sum_{i=1}^{n-1}(-1)^{i}\binom{n-1}{i}e^{-i\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \\ &\times\ e^{-\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \\ &= n\left\{\alpha\beta(1+\alpha x)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x^{\gamma})^{\theta-1}\right\}\sum_{i=1}^{n-1}(-1)^{i}\binom{n-1}{i}e^{-(i+1)\left[(1+\alpha x)^{\beta}+(1+\lambda x^{\gamma})^{\theta}-2\right]} \end{aligned}$$

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4.4 Maximum Likelihood Estimation

This study employed the the maximum likelihood estimation technique to obtain estimators for the unknown parameters of the NHGPW distribution. MLE obtain estimates of the parameter values that maximises the likelihood function. The likelihood function is defined as;

$$L=\prod_{i=1}^{n}f(x). \tag{4.42}$$

For the NHGPW distribution with PDF given in equation (4.12), the likelihood function is given as;

$$L=\prod_{i=1}^{n}\left\{\alpha\beta(1+\alpha x_i)^{\beta-1}+\lambda\gamma\theta x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}\right\}.e^{-\left[(1+\alpha x_i)^{\beta}+(1+\lambda x_i^{\gamma})^{\theta}-2\right]}. \tag{4.43}$$

we obtain the score function by taking logarithm of equation (4.49)

$$l=\sum_{i=1}^{n}log\left\{\alpha\beta(1+\alpha x_i)^{\beta-1}+\lambda\gamma\theta x^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}\right\}-\sum_{i=1}^{n}\left[(1+\alpha x_i)^{\beta}+(1+\lambda x_i^{\gamma})^{\theta}-2\right].$$

To obtain the MLE of the parameters of the NHGPW distribution, we maximize the score function by taking the its first derivative and equating the derivatives to zero. We then make the parameter of interest the subject.

These MLE parameter estimates are given as;

$$\frac{\partial l}{\partial \alpha} = -\sum_{i=1}^{n} \beta x_i (1+\alpha x_i)^{\beta-1} + \sum_{i=1}^{n} \frac{\alpha(\beta-1)\beta x_i (1+\alpha x_i)^{\beta-2} + \beta(1+\alpha x_i)^{\beta-1}}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{4.44}$$

$$\frac{\partial l}{\partial \beta} = -\sum_{i=1}^{n} log[1+\alpha x_i](1+\alpha x_i)^{\beta} + \sum_{i=1}^{n} \frac{\alpha(1+\alpha x_i)^{\beta-1} + \alpha\beta log[1+\alpha x_i](1+\alpha x_i)^{\beta-1}}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{4.45}$$

$$\frac{\partial l}{\partial \theta} = -\sum_{i=1}^{n} log[1+\lambda x_i^{\gamma}](1+\lambda x_i^{\gamma})^{\theta} + \sum_{i=1}^{n} \frac{\gamma\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1} + \gamma\theta\lambda log[1+\lambda x_i^{\gamma}] x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{4.46}$$

$$\frac{\partial l}{\partial \lambda} = -\sum_{i=1}^{n} \theta x_i^{\gamma}(1+\lambda x_i^{\gamma})^{\theta-1} + \sum_{i=1}^{n} \frac{\gamma(\theta-1)\theta\lambda x_i^{2\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-2} + \gamma\theta x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{4.47}$$

$$\frac{\partial l}{\partial \gamma} = -\sum_{i=1}^{n} \theta\lambda log[x_i] x_i^{\gamma}(1+\lambda x_i^{\gamma})^{\theta-1}.$$

$$+\sum_{i=1}^{n} \frac{(\gamma(\theta-1)\theta\lambda^2 log[x_i] x_i^{2\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-2} + \theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1} + \gamma\theta\lambda log[x_i] x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1})}{\alpha\beta(1+\alpha x_i)^{\beta-1} + \gamma\theta\lambda x_i^{\gamma-1}(1+\lambda x_i^{\gamma})^{\theta-1}}. \tag{4.48}$$

Therefore, the variance-covariance matrix of the parameters is given as;

$$A^{-1} = \begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\gamma\beta} & \frac{\partial^2 l}{\partial\alpha\gamma\theta} & \frac{\partial^2 l}{\partial\alpha\gamma\lambda} & \frac{\partial^2 l}{\partial\alpha\gamma\gamma} \\ & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\gamma\theta} & \frac{\partial^2 l}{\partial\beta\gamma\lambda} & \frac{\partial^2 l}{\partial\beta\gamma\gamma} \\ & & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\theta\gamma} & \frac{\partial^2 l}{\partial\lambda\gamma\gamma} \\ & & & \frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\theta\gamma\gamma} \\ & & & & \frac{\partial^2 l}{\partial\gamma^2} \end{bmatrix}$$

4.5 Monte Carlo Simulation

Monte Carlo simulations were conducted to assess the performance of the maximum likelihood estimators for the parameters of the NHGPW distribution. Five different combinations of parameter values of this distribution were specified and its quantile function then used to generate five different random samples of sizes, $n = 40, 80, 120, 160, 200$. These were then used to obtain the maximum likelihood estimates of the parameters of the distribution. The simulation was replicated for N=1000 times. The average bias (ABias) and mean square error (MSE) were calculated for the estimators of the parameters of the NHGPW distribution. The results of the simulation are shown in Table 4.4. The results showed that, as the sample size increases the maximum likelihood estimates of the parameters of the NHGPW distribution converges to the true parameter value since the mean square errors decay to zero and the biases of each parameter also decrease.

Table 4.4: **Monte Carlo Simulation Results: ABias and MSE for the Parameters of the NHGPW distribution**

n	Parameter value					ABiase					MSE				
	α	β	λ	θ	γ	α	β	λ	θ	γ	α	β	λ	θ	γ
40	0.4	0.4	0.5	0.3	2.7	0.43	0.21	0.28	0.150	0.35	0.24	0.06	0.10	0.07	0.17
80	0.4	0.4	0.5	0.3	2.7	0.41	0.20	0.26	0.10	0.34	0.23	0.06	0.08	0.04	0.16
120	0.4	0.4	0.5	0.3	2.7	0.41	0.20	0.20	0.10	0.34	0.22	0.05	0.06	0.03	0.16
160	0.4	0.4	0.5	0.3	2.7	0.40	0.20	0.19	0.08	0.34	0.22	0.06	0.06	0.02	0.16
200	0.4	0.4	0.5	0.3	2.7	0.39	0.20	0.17	0.07	0.33	0.20	0.05	0.05	0.02	0.15
40	0.3	0.5	0.7	0.5	2.5	0.53	0.28	0.34	0.41	0.52	0.35	0.10	0.13	0.20	0.28
80	0.3	0.5	0.7	0.5	2.5	0.41	0.28	0.28	0.35	0.53	0.24	0.10	0.10	0.15	0.28
120	0.3	0.5	0.7	0.5	2.5	0.36	0.28	0.27	0.32	0.54	0.19	0.07	0.08	0.13	0.28
160	0.3	0.5	0.7	0.5	2.5	0.32	0.23	0.23	0.29	0.53	0.16	0.01	0.07	0.11	0.28
200	0.3	0.5	0.7	0.5	2.5	0.30	0.27	0.20	60.28	0.52	0.14	0.01	0.06	0.10	0.27
40	0.32	0.5	0.8	0.6	2.7	0.62	0.35	0.28	0.39	0.66	0.41	0.13	0.09	0.16	0.44
80	0.32	0.5	0.8	0.6	2.7	0.59	0.34	0.25	0.38	0.66	0.38	0.13	0.07	0.15	0.43
120	0.32	0.5	0.8	0.6	2.7	0.57	0.34	0.25	0.38	0.66	0.36	0.13	0.07	0.15	0.44
160	0.32	0.5	0.8	0.6	2.7	0.57	0.34	0.24	0.38	0.66	0.36	0.13	0.07	0.15	0.43
200	0.32	0.5	0.8	0.6	2.7	0.57	0.33	0.24	0.38	0.66	0.36	0.13	0.06	0.15	0.43
40	0.9	1.1	1.0	1.0	1.5	0.58	0.50	0.83	0.60	0.93	0.48	0.34	0.76	0.49	1.69
80	0.9	1.1	1.0	1.0	1.5	0.56	0.45	0.81	0.57	0.77	0.46	0.30	0.74	0.44	1.31
120	0.9	1.1	1.0	1.0	1.5	0.54	0.42	0.80	0.55	0.71	0.43	0.29	0.72	0.42	1.18
160	0.9	1.1	1.0	1.0	1.5	0.49	0.43	0.78	0.51	0.67	0.37	0.27	0.70	0.38	1.07
200	0.9	1.1	1.0	1.0	1.5	0.48	0.41	0.76	0.51	0.62	0.36	0.25	0.68	0.37	0.96
40	0.9	1.1	1.0	1.0	2.5	0.69	0.56	0.72	0.67	1.00	0.61	0.39	0.59	0.58	1.28
80	0.9	1.1	1.0	1.0	2.5	0.61	0.60	0.71	0.63	0.83	0.50	0.44	0.58	0.53	0.93
120	0.9	1.1	1.0	1.0	2.5	0.56	0.62	0.71	0.61	0.75	0.44	0.46	0.58	0.50	0.81
160	0.9	1.1	1.0	1.0	2.5	0.59	0.63	0.69	0.62	0.74	0.48	0.47	0.56	0.51	0.78
200	0.9	1.1	1.0	1.0	2.5	0.56	0.63	0.65	0.63	0.70	0.38	0.48	0.51	0.52	0.70

4.6 Applications

The NHGPW was applied on two sets of data (Kevlar 49/epoxy strands failure rate data and Aircraft Windshield failure rate data) and various goodness-of-fit analyses performed. The goodness-of-fit of the NHGPW distribution was compared with some of its sub-models and other five parameter distributions for systems in series. Thus; the NH distribution, the GPW distribution, exponential-exponential (EE) distribution, beta modified Weibull (BetaMW) distribution, beta Weibull Poisson (BetaWP) distribution, Gamma log-logistic Weibull (GLLoGW) distribution, Weibull NH (WNH) distribution, Kumaraswamy log-logistic Weibull (KLLoGW) distribution, Exponentiated Generalised Poisson inverse Exponential (EGPIE) distribution and the Exponentiated Generalised Geometric inverse exponential (EGGIE) distribution. The comparison was done using the Kolmogorov Smirnov statistic, Cramér-Von Mises statistic, Anderson- Darling statistic, log-likelihood and model selection criteria such as the AIC, AICc and BIC. The PDFs and CDFs of the NH, PGW, EE, EGGIE, KLLoGW, GLLoGW, BetaMW, BetaWP distributions are respectively given as;

$$f_{NH}(t) = \alpha\beta(1+\alpha t)^{\beta-1}e^{(1-(1+\alpha t)^{\beta})}, \quad \alpha,\beta > 0, t > 0,$$

$$f_{GPW}(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}, \quad t > 0, \lambda,\gamma,\theta > 0,$$

$$f_{EE}(t) = (\alpha+\lambda)e^{-[\alpha t+\lambda t]}, \alpha > 0, \lambda > 0,$$

$$f_{EGGIE}(t) = \frac{(1-\lambda)\lambda\gamma c d t^{-2}e^{-\gamma t^{-1}}(1-e^{-\gamma t^{-1}})^{d-1}(1-(1-e^{-\gamma t^{-1}})^{d})^{c-1}}{[1-\lambda[1-(1-(1-e^{-\gamma t^{-1}})^{d}]^{c}]^{2}},$$

$0 < \lambda < 1,\ \lambda,\gamma,c,d,t > 0,$

$$f_{KLLoGW} = ab\left(1-(1+t^{c})^{-1}e^{-\alpha t^{\beta}})^{\alpha-1}\right)\left(1-(1-(1+t^{c})^{-1}e^{-\alpha t^{\beta}})^{a}\right)^{b-1}(1+t^{c})^{-2}e^{-\alpha t^{\beta}},$$

$a, b, c, \alpha, \beta, t > 0,$

$$f_{GLLoGW}(t) = \frac{1}{\Gamma(a)\theta^{a}}(1+t^{c})^{-1}e^{-\alpha t^{\beta}}\left[1-(1+t^{c})^{-1}ct^{c-1}+\alpha\beta t^{\beta-1}\right]\left(-\log(1-(1+t^{c})^{-1})e^{-\alpha t^{\beta}}\right)$$

$$\times \left(1 - (1 + t^c)^{-1} e^{-\alpha t^{\beta}}\right),$$

$c, \alpha, \beta, a, \theta, t > 0,$

$$f_{betaMW}(t) = \frac{\alpha t^{\gamma-1}(\gamma + \lambda t)e^{\lambda t}}{B(a,b)} e^{-bat^{\gamma}}(1 - e^{-\alpha t^{\gamma} e^{\lambda t}})^{\alpha-1},$$

$\alpha, \gamma, \lambda, a, b, t > 0,$

$$f_{betaWP}(t) = \frac{\alpha\beta\lambda t^{\alpha-1} e^{\lambda e^{\beta t^{\alpha}} - \lambda - \beta t^{\alpha}} (e^{\lambda} - 1)^{2-a-b} (e^{\lambda} - e^{\lambda e^{\beta t^{a}}})^{a-1} (e^{\lambda e^{\beta t^{\alpha}}} - 1)^{b-1}}{B(a,b)(1 - e^{-\lambda})},$$

$\alpha, \beta, \lambda, a, b, t > 0,$

4.6.1 Application I: Kevlar 49/epoxy Strands Data

The first data set consist of 101 observations representing the failure time (in hours) of Kevlar 49/epoxy strands subjected to constant sustained pressure at 90 percent stress level. Table 6.1 in Appendix A gives this data set.

Table 4.5 displays the descriptive statistics for the Kevlar 49/epoxy Strands failure time data. It is seen that the data set is positively skewed and leptokurtic in nature since the skewness value is positive and the kurtosis value greater than three. This implies that, the distribution of this data set is more peaked as compared to the normal distribution and majority of the data points are clustered at the lower side of the distribution with a long tail to the right. Since the PDF of the NHGPW distribution can be positively skewed, it implies this distribution can be applied to the Kevlor 49/expoxy strands data.

Table 4.5: **Descriptive Statistics of Kevlar 49/epoxy Strands failure data**

Statistic	Mean	St.Dev	Coeff. Variation	Median	Kurtosis	Skewness
Value	1.025	1.119	1090.220	0.800	14.470	3.080

The TTT transformed plot was employed to explore the shape of the hazard function of the data set before the main application is done. The TTT transformed plot as shown in Figure 4.5 is first convex in shape, followed by a concave shape and then another convex shape which indicate that the hazard function of the Kevlar 49/epoxy data set exhibits a modified bathtub shape. The hazard function of the NHGPW distribution can be modified bathtub hence the NHGPW distribution is applicable this data.

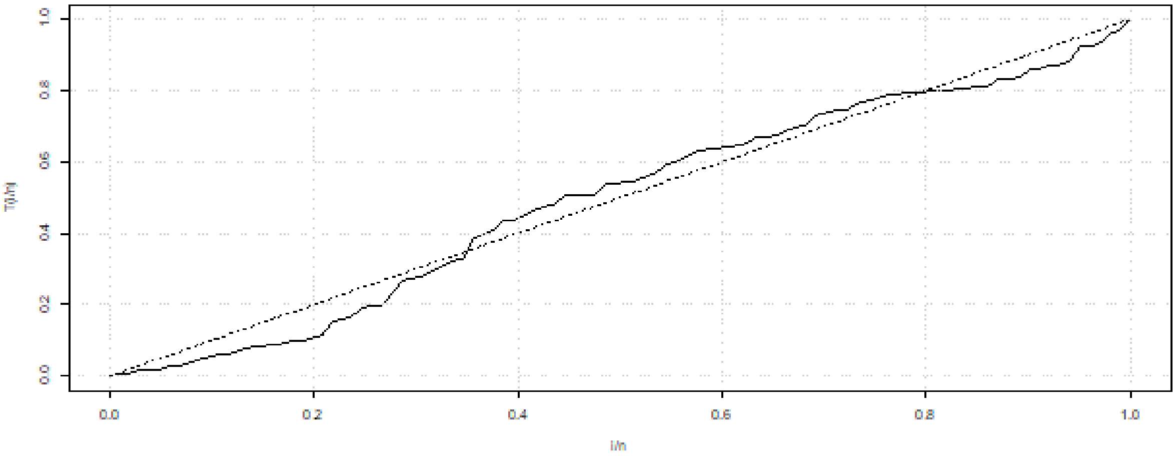

Figure 4.5: **TTT plot of Kevlar 49/epoxy data set**

The detailed parameter estimates of the NHGPW distribution and the competitive distributions considered for the Kevlar 49/epoxy data are shown in Table 4.6. Using the standard errors of the NHGPW distribution, it is seen that all the parameters are significant at 5 percent significance level since their standard errors are less than half of their parameter estimates.

Table 4.6: **Maximum Likelihood Parameter Estimates of the Kevlar 49/epoxy Data**

Distribution	Parameter estimates and Standard errors				
NHGPW	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
	3.435	0.414	5.638	0.007	114.570
	(1.618)	(0.155)	(0.022)	(0.002)	(0.000)
NH	1.145	0.693			
	(0.431)	(0.189)			
GPW			0.593	0.762	1.308
			(0.465)	(0.125)	(0.645)
Exp-Exp	0.087		1.389		
	(0.049)		(0.049)		
BetaWP	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
	0.075	8.395	0.825	1.070	374.510
	(0.018)	(0.390)	(0.276)	(0.201)	(0.005)
GLLoGW	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\theta}$
	0.237	0.259	0.965	4.396	0.140
	(0.297)	(0.373)	(0.374)	(10.719)	(0.333)
BetaMW	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$
	108.860	25.631	1.663	0.053	0.034
	(0.000)	(0.001)	(0.279)	(0.008)	(0.009)
KLLoGW	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$
	1733.340	0.494	4.248	8.474	8.474
	(6857.670)	(0.175)	(1.227)	(0.0171)	(3.877)
EGPIE	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	
	26.062	7.320	0.175	0.002	
	(0.009)	(1.770)	(0.175)	(0.002)	
EGGIE	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	
	0.664	20.525	0.498	0.002	
	(0.241)	(0.005)	(0.137)	(0.000)	
WNH	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	
	0.440	0.263	11.643	0.915	
	(0.497)	(0.235)	(18.644)	(0.095)	

Table 4.7 presents the goodness-of-fit measures and the information criteria for the NHGPW distribution and the competitive distributions. As compared to the other distributions, the developed NHGPW distribution has the highest log-likelihood value with the smallest values of the Kolmogorov Smirnov (KS), Anderson-Darling (AD), Crame′r-Von Mises (CVM) statistics. By using the model selection criteria, the NHGPW distribution has the smallest AIC, AICc, and BIC values as compared to the other distributions. These indicate that, the NHGPW distribution provides a better fit to the Kevlar 49/epoxy data set as compared to the existing distributions.

Table 4.7: **Goodness of fit and Information Criteria of Kevlar 49/epoxy data**

Distribution	LL	$-2\log L$	AIC	AICc	BIC	CVM	AD	KS(p-value)
NHGPW	−95.012	190.024	199.89	200.580	213.253	0.088	0.124	0.0691(0.721)
NH	−103.340	206.683	210.683	211.373	215.913	0.204	1.139	0.082(0.005*)
Exp-Exp	−103.480	206.959	210.959	210.730	216.189	0.181	1.028	0.205(0.000*)
GPW	−102.800	205.601	245.124	245.814	252.969	0.1716	0.985	0.205(0.000*)
BetaWP	−101.200	202.400	212.040	212.700	225.120	0.114	0.702	0.150(0.000*)
GLLoGW	−102.050	204.100	214.010	214.600	227.100	0.132	0.800	0.132(0.000*)
BetaMW	−103.650	207.300	217.300	217.900	230.38	0.196	1.119	0.080(0.000*)
KLLoGW	−95.550	191.100	201.100	201.700	214.100	0.212	0.1534	1.000(0.000*)
EGPIE	−116.660	233.320	241.314	241.947	251.774	0.738	0.158	0.182(0.000*)
EGGIE	−140.090	280.180	288.170	288.802	298.631	1.386	0.133	0.237(0.000*)
WNH	−103.010	206.025	214.025	214.715	224.485	0.195	1.097	0.090(0.383)

The likelihood ratio (LR) test was further performed to compare the fitness of the NHGPW distribution with its sub-distributions. The results as shown in Table 4.8 revealed that, the fit of the NHGPW is significantly different from its sub-distributions since significant test statistics were obtained for all of them (corresponding p-values are all less 0.05 significance level). The results also showed that, the NHGPW distribution fits well to the Kevlar 49/epoxy data than its sub-distributions.

Table 4.8: **LR Test Statistic for Kevlar 49/epoxy Data**

Distribution	Deviance	LRT	p-value
NH	206.680	7.506	0.044*
GPW	205.600	6.424	0.040*
Exp-Exp	206.960	7.782	0.000*

The asymptotic variance covariance matrix of the parameter estimates of the NHGPW distribution for the Kevlar 49/epoxy data is given by;

$$A^{-1} = \begin{bmatrix} 2.617 & -0.349 & -0.052 & 2.011\times10^{-3} & -5.299\times10^{-4} \\ -0.349 & 2.409\times10^{-2} & 3.357\times10^{-3} & -1.493\times10^{-4} & 3.407\times10^{-5} \\ -0.052 & 3.357\times10^{-3} & 5.027\times10^{-4} & -1.937\times10^{-5} & 5.102\times10^{-6} \\ 2.011\times10^{-3} & -1.493\times10^{-4} & -1.937\times10^{-5} & 2.185\times10^{-6} & -1.965\times10^{-7} \\ -5.299\times10^{-4} & 3.407\times10^{-5} & 5.102\times10^{-6} & -1.965\times10^{-7} & 5.178\times10^{-8} \end{bmatrix}$$

The approximate 95 percent confidence interval (CI) for the five parameters of the NHGPW distribution are; $\alpha : [0.2647; 6.6057], \beta : [0.1099; 0.7183], \lambda : [5.5941; 5.6819], \theta : [00036; 0.0094]$; and $\gamma : [114.5696; 114.570$. The estimated CI of the parameters of the NHGPW distribution also showed that, its parameters were all significant at 5 percent significance level since their estimated confidence intervals do not contain zero.

Figure 4.6 gives the plot of the empirical CDF, the CDF of the NHGPW distribution and the CDFs of the comparison distributions. As it is seen from the figure, the NHGPW distribution fits better to this data set as compared to the considered models since its CDF approximates the empirical CDF.

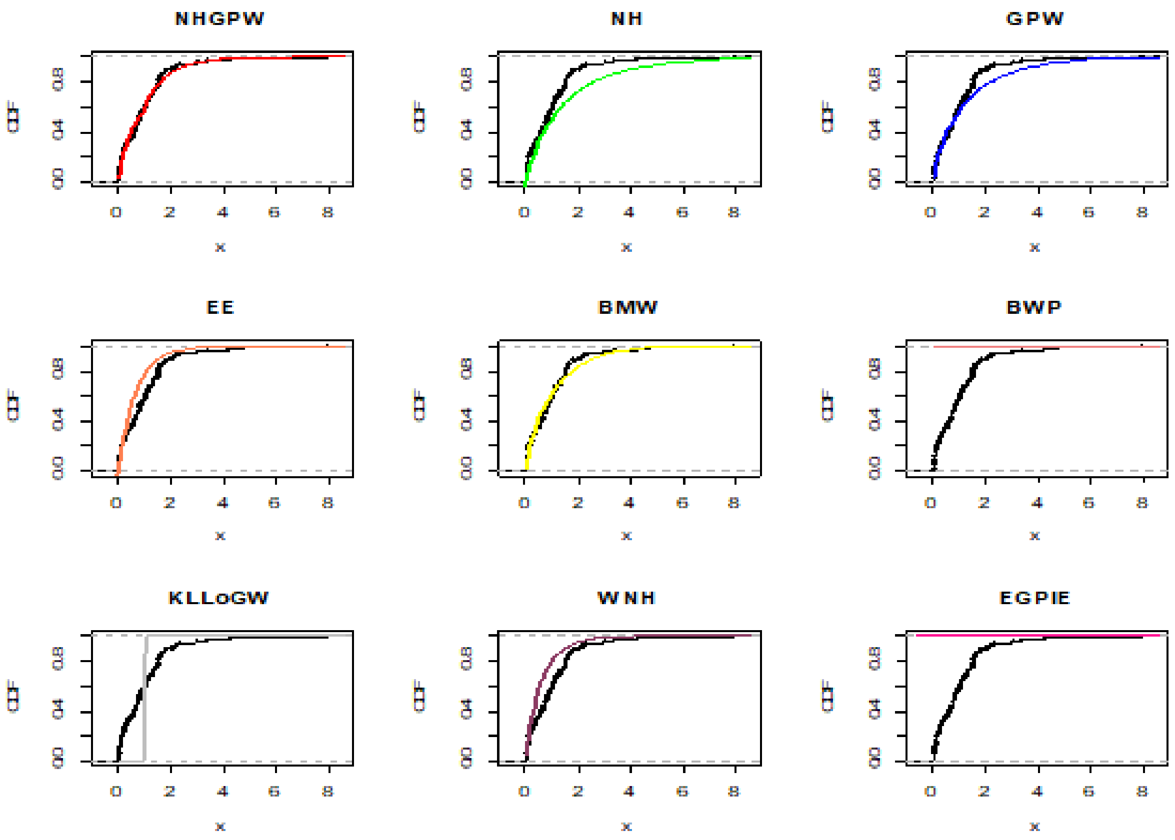

Figure 4.6: **Plots of CDFs of the Kevlar 49/epoxy data set**

4.6.2 Application II: Air Craft Windshield Failure Data Set

The NHGPW was also applied on failure times data of 84 aircraft windshield. To make a comparison between the NHGPW distribution and the other distributions, the estimated Log-likelihood, AIC, AICc, BIC, Crame$'$r Von Mises (CVM), Anderson-Darling (AD), Kolmogorov Smirnov (KS) goodness-of-fit statistics were calculated for all the competitive distributions. The NHGPW distribution was compared with the NH distribution, the GPW distribution, EE distribution, BetaMW, WNH and KLLoGW distributions. This data is given in Table 6.2 in Appendix A.

Table 4.9 gives the descriptive statistics for the failure time data for the 84 aircraft windshield. It is seen that the data set is positively skewed and platikurtic in nature since the skewness value is positive and the kurtosis value is less than three.

Table 4.9: **Descriptive Statistics of Air Craft Windshield Failure data**

Statistic	Mean	St.Dev	CV	Median	Kurtosis	Skewness
Value	2.563	1.113	43.440	2.385	-0.090	0.090

The TTT transform plot of the failure rate data of the 84 aircraft windsheild shown in Figure 4.7 indicates that, its hazard function has an increasing failure rate.

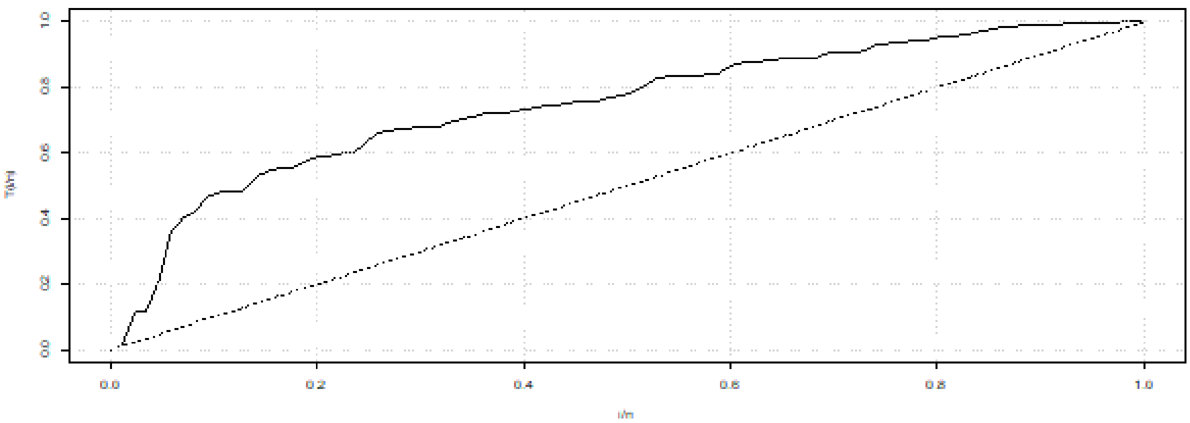

Figure 4.7: **TTT Plot of 84 Aircraft Windshield data set**

The parameter estimates and standard errors of the NHGPW distribution and the candidate distributions for the aircraft windshield failure rate data are shown in Table 4.10. By using the standard errors of the NHGPW distribution, parameters α, β, λ, and γ are significant at 5 percent significant level since their standard errors are less than half of their parameter estimates whiles θ is insignificant since its standard error is greater than half of the parameter estimate.

Table 4.10: **Maximum Likelihood Parameter Estimate for Aircraft Windshield Failure Data**

Distribution	Parameter Estimates and Standard errors				
NHGPW	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
	101.826	0.005	0.003	2.044	2.303
	(0.009)	(0.000)	(0.000)	(3.009)	(0.594)
NH	0.008	33.695			
	(0.000)	(0.000)			
GPW			0.010	1.757	10.051
			(0.002)	(0.172)	(0.001)
Exp-Exp	0.345		0.045		
	(0.021)		(0.021)		
BetaMW	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$
	0.286	0.009	6.224	2.543	0.057
	(0.184)	(0.003)	(0.012)	(1.062)	(0.341)
KLLoGW	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$
	7.934	12.501	0.129	0.15	1.286
	(5.499)	(43.587)	(0.143)	(0.114)	(0.651)
WNH	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	
	0.562	4.22	0.023	1.088	
	(0.226)	(9.357)	(0.043)	(0.547)	

The goodness-of-fit and information criteria for the competitive distributions are presented in Table 4.11. Among the competitive distributions, the developed NHGPW was shown to the best distribution for the aircraft windshield failure data set since it has the minimum value of all the information criteria (AIC=264.059, AICc=264.828 and BIC=276.272) as well as the minimum value of the goodness-of-fit statistics (K-S=0.85, AD=0.5, CVM=0.062) with the largest log likelihood value (LL=-127.230).

Table 4.11: **Goodness of Fit and Information Criteria of Air Craft Windshield Data**

Distribution	LL	$-2\log L$	AIC	AICc	BIC	CVM	AD	K-S(p-value)
NHGPW	-127.230	254.081	264.059	264.828	276.272	0.062	0.534	0.085(0.571)
NH	-145.550	291.100	295.099	295.248	299.985	0.082	0.610	0.258(0.000*)
Exp-Exp	-164.990	329.975	333.975	334.123	338.861	0.166	1.398	0.303(0.000*)
GPW	-128.960	256.946	686.572	686.872	693.900	0.307	2.294	0.915(0.000*)
BetaMW	-128.260	254.260	264.517	265.286	277.730	3.129	17.136	0.682(0.000*)
KLLoGW	-127.990	255.971	265.971	266.740	278.184	0.077	0.545	0.086(0.552)
WNH	-128.180	256.355	264.355	264.861	274.125	0.105	0.692	0.088(0.527)

The LR test results shown in Table 4.12 indicate that, the fit of the NHGPW is significantly different from the parent distribution (NH and GPW distributions) since significant test statistics were obtained for all of them (corresponding p-values are all less 0.05 significance level).

Table 4.12: **LR Test Statistic for Air Craft Windshield Data**

Distribution	Deviance	LRT	p-value
NH	291.100	36.018	0.000*
GPW	257.930	17.064	0.040*

The asymptotic variance covariance matrix for the parameter estimates of the NHGPW distribution for the air craft windshield failure data is given by;

$$A^{-1} = \begin{bmatrix} 8.126\times10^{-5} & 1.324\times10^{-5} & 3.415\times10^{-4} & -2.712\times10^{-2} & 4.981\times10^{-3} \\ 1.324\times10^{-5} & 4.076\times10^{-6} & 4.283\times10^{-5} & -4.358\times10^{-3} & 1.060\times10^{-3} \\ 3.415\times10^{-4} & 4.283\times10^{-5} & 1.690\times10^{-6} & -1.145\times10^{-1} & 1.869\times10^{-2} \\ -2.712\times10^{-2} & -4.358\times10^{-3} & -1.145\times10^{-1} & 9.052 & -1.651 \\ 4.981\times10^{-3} & 1.060\times10^{-3} & 1.869\times10^{-2} & -1.651 & 0.352 \end{bmatrix}$$

For the aircraft windshield data, the approximate 95 percent CI of the five parameters of the NHGPW distribution are; $\alpha : [101.808; 101.844]$, $\beta : [0.001; 0.009]$, $\lambda : [0.0005; 0.006]$, $\theta : [-3.853; 7.941]$; and $\gamma : [1.140; 3.466]$.

The plot of the empirical CDF, the CDF of the NHGPW distribution and the CDFs of the competitive distributions are shown in Figure 4.8. From the plots, the CDF of the

NHGPW distribution approximates the empirical CDF of the aircraft windshield failure failure data set hence provides a better fit as compared to the other distributions considered.

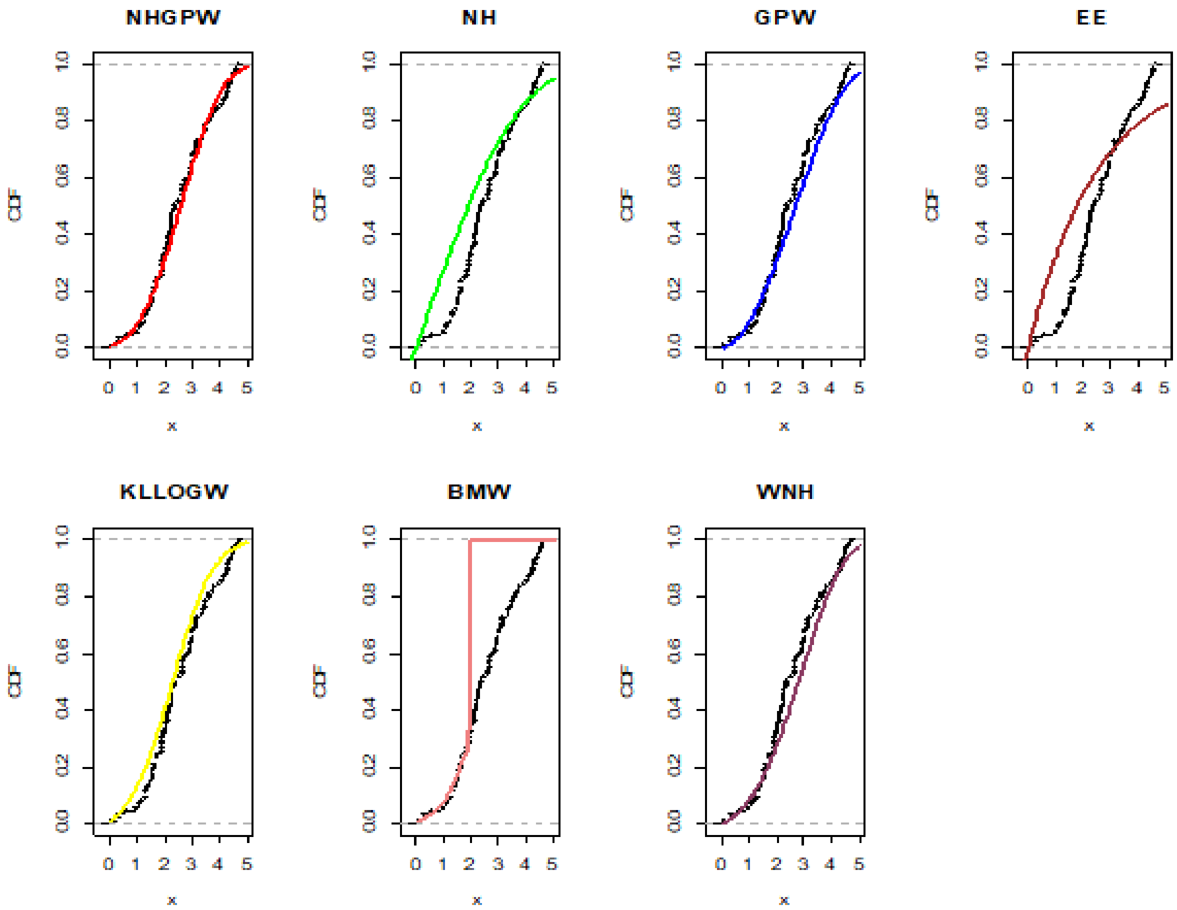

Figure 4.8: **Empirical CDF and CDF plots of Aircraft Windshield Failure data set**

CHAPTER 5

The POWER SERIES GENERALISED POWER WEIBULL CLASS OF DISTRIBUTIONS

5.1 Introduction

In this section, the Power Series Generalised Power Weibull (PGPW) class of distributions was developed for modelling failure rate from system with subsystems connected in series. This class of distributions was obtained by compounding the generalised Power Weibull and the power series family of distributions.

5.2 The Power Series Generalised Power Weibull Class Of Distributions

Consider N to be a discrete random variable from the power series distribution (truncated at zero). N gives the number of failures of system with independent subsystem functioning in series at a given point in time. Hence N has a probability mass function (PMF);

$$P(N = n) = \frac{a_n \alpha^n}{C(\alpha)}, n = 1, 2, ... \tag{5.1}$$

$a_n > 0$, $C(\alpha) = \sum_{n=1}^{\infty} a_n \alpha^n$ and $\alpha \in (0, s)$. a_n is the coefficient of the power series, $C(\alpha)$ is the generating function and s is the parameter space.

Assume $T_1, T_2, ..., T_N$ represents the lifetimes failures associated with this system of independent and identically distributed continuous random variables following the generalised power Weibull distribution (GPW $(\lambda, \theta, \gamma)$). The CDF of T is given by;

$$F(t) = 1 - e^{[1-(1+\lambda t^{\gamma})^{\theta}]}. \tag{5.2}$$

with Survival function given by;

$$s(t) = e^{[1-(1+\lambda t^{\gamma})^{\theta}]}. \tag{5.3}$$

T_i gives the time to failure of the i series subsystem. Since the subsystems are in series and all subsystems must be working for the systems success, T_1 is defined by;

$$T_1 = min\left\{T_1, T_2, ..., T_N\right\}.$$

Then the conditional cumulative distribution function of $T_1 \mid N = n$ is given as;

$$\begin{aligned} F_{T(1)|N=n}(t) &= 1 - \prod_{i=1}^{n}[1 - F_i(t)] \\ &= 1 - [s(t)]^n. \end{aligned}$$

Hence,

$$F_{T(1)|N=n}(t) = 1 - \left[e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]^n, \ \ t > 0. \tag{5.4}$$

Also, the marginal CDF of T_1 which gives the CDF of the new distribution (PGPW class of distribution) is given as;

$$\begin{aligned} F(t;\alpha,\lambda,\gamma,\theta) &= \sum_{n=1}^{\infty} P(N=n)F_{T(1)|N=n}(t) \\ &= \sum_{n=1}^{\infty}\left[1 - (e^{[1-(1+\lambda t^{\gamma})^{\theta})]})^n\right] \times \frac{a_n\alpha^n}{C(\alpha)} \\ &= \sum_{n=1}^{\infty}\left[1 - e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}\right] \times \frac{a_n\alpha^n}{C(\alpha)} \\ &= \sum_{n=1}^{\infty}\frac{a_n\alpha^n}{C(\alpha)} - \sum_{n=1}^{\infty}\frac{a_n\alpha^n}{C(\alpha)}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]} \\ &= \frac{\sum_{n=1}^{\infty} a_n\alpha^n}{\sum_{n=1}^{\infty} a_n\alpha^n} - \sum_{n=1}^{\infty}\frac{a_n\alpha^n}{C(\alpha)}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]} \\ &= 1 - \frac{\sum_{n=1}^{\infty} a_n\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]^n}{C(\alpha)} \end{aligned}$$

Since $C(\alpha) = \sum_{n=1}^{\infty} a_n \alpha^n$,

$$F(t;\alpha,\lambda,\gamma,\theta) = 1 - \frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}, \;\; t > 0, \alpha > 0, \gamma > 0, \theta > 0. \qquad (5.5)$$

The PDF of the PGPW class of distributions obtained by differentiating its CDF is given as;

$$f(t) \;=\; \alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \frac{C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}, t > 0, \qquad (5.6)$$

where $\alpha > 0, \lambda > 0$ are scales parameters and $\gamma > 0, \theta > 0$ are shape parameters. The survival function $s(t)$ of the PGPW family of distributions is;

$$\begin{aligned} s(t) &= 1 - F(t) \\ &= 1 - 1 + \frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}, \end{aligned}$$

$$s(t) = \frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}. \qquad (5.7)$$

The hazard $(h(t))$ function is expressed as;

$$h(t) \;=\; \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times \frac{C'\left[\alpha e^{1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}}{\frac{C\left[\alpha e^{[(1-(1+\lambda t^{\gamma})^{\theta})]}\right]}{C(\alpha)}}$$

$$h(t) = \alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times \frac{C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}. \qquad (5.8)$$

and the reversed hazard function is given;

$$\begin{aligned} R(t) &= \frac{f(t)}{F(t)} \\ &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda)^{\theta})} \times \frac{C'\left[\alpha e^{1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}}{1 - \frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}} \end{aligned}$$

$$R(t) = \alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda)^{\theta})}\frac{C'\left[\alpha e^{1-(1+\lambda t^{\gamma})^{\theta]}}\right]}{C(\alpha) - C\left[\alpha e^{1-(1+\lambda t^{\gamma})^{\theta]}}\right]}. \tag{5.9}$$

From the PGPW class of distributions, a number of sub-distributions can be developed. These include; the power series Weibull distribution if $\theta = 1$ with CDF and PDF given as;

$$F_{PW}(t) = 1 - \frac{C\left[\alpha e^{-\lambda t^{\gamma}}\right]}{C(\alpha)}. \tag{5.10}$$

and

$$f_{PW}(t) = \alpha\lambda\gamma t^{\gamma-1}e^{-\lambda t^{\gamma}} \times \frac{C'\left[\alpha e^{-\lambda t^{\gamma}}\right]}{C(\alpha)}. \tag{5.11}$$

If $\theta = 1$, $\gamma = 1$, we have the power series exponential distribution with CDF and PDF given by;

$$F_{PE}(t) = 1 - \frac{C\left[\alpha e^{-\lambda t}\right]}{C(\alpha)}. \tag{5.12}$$

and

$$f_{PE}(t) = \alpha\lambda e^{-\lambda t}\frac{C'\left[\alpha e^{-\lambda t}\right]}{C(\alpha)}. \tag{5.13}$$

if $\gamma = 1$, we have the power series NH distribution with CDF and PDF given by;

$$F_{PNH}(t) = 1 - \frac{C\left[\alpha e^{(1-(1+\lambda t)^{\theta})}\right]}{C(\alpha)}. \tag{5.14}$$

$$f_{PNH}(t) = \alpha\lambda\theta(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t))^{\theta}}\frac{C'\left[\alpha e^{(1-(1+\lambda t)^{\theta})}\right]}{C(\alpha)}. \tag{5.15}$$

Proposition 5.1. For $\alpha \to 0$, the GPW is a limiting distribution of the PGPW class of distributions.

Proof. Using the CDF of the PGPW class of distributions, we obtain the limits as;

$$\lim_{\alpha\to 0}(F(t)) \quad = \quad 1 - \lim_{\alpha\to 0}\frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}.$$

using $C(\alpha) = \sum_{n=1}^{\infty} a_n\alpha^n$, we have;

$$\lim_{\alpha\to 0} F(t) = 1 - \lim_{\alpha\to 0}\frac{\sum_{n=1}^{\infty} a_n\alpha^n e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}}{\sum_{n=1}^{\infty} a_n\alpha^n}.$$

Using the concept of the L' Hoptal rule to simplify, we obtain

$$\begin{aligned}
\lim_{\alpha\to 0} F(t) &= 1-\lim_{\alpha\to 0}\frac{\sum_{n=1}^{\infty} na_n\alpha^{n-1}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}}{\sum_{n=1}^{\infty} na_n\alpha^{n-1}}\\
&= 1-\lim_{\alpha\to 0}\frac{\left[a_1 e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]+\sum_{n=2}^{\infty} a_n\alpha^{n-1}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}}{a_1+\sum_{n=2}^{\infty} na_n\alpha^{n-1}}\\
&= 1-\frac{a_1 e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{a_1}\\
&= 1-e^{[1-(1+\lambda t^{\gamma})^{\theta}]}.
\end{aligned}$$

$1-e^{[1-(1+\lambda t^{\gamma})^{\theta}]}$ is the CDF of the GPW distribution.

Lemma 5.1. The CDF of the PGPW class of distributions is well defined.

Proof. For the CDF of a continuous distribution to be well defined, then the following must be true;

$$\left\{\begin{array}{ll} \text{t}\longrightarrow\infty, & F(x)\rightarrow 1\\ \text{t}\longrightarrow -\infty, & F(x)\rightarrow 0 \end{array}\right\}.$$

For the PGPW class of distributions, if $t\rightarrow\infty$ we have

$$\begin{aligned}
\lim_{t\to\infty} F(t) &= \lim_{t\to\infty}\left[1-\frac{C\left[\alpha e^{[1-(1+\lambda(\infty)^{\gamma})^{\theta}]}\right]}{C(\alpha)}\right]\\
&= \left[1-\frac{C[\alpha e^{-\infty}]}{C(\alpha)}\right]\\
&= \left[1-\frac{C\left[\alpha(0)\right]}{C(\alpha)}\right]\\
&= 1-0\\
&= 1.
\end{aligned}$$

Also, for $t \to 0$

$$\begin{aligned}\lim_{t\to 0} F(t) &= \lim_{t\to 0}\left[1 - \frac{C\left[\alpha e^{[1-(1+\lambda(0)^{\gamma})^{\theta}]}\right]}{C(\alpha)}\right]\\ &= \left[1 - \frac{C[\alpha e^{0}]}{C(\alpha)}\right]\\ &= \left[1 - \frac{C(\alpha)}{C(\alpha)}\right]\\ &= 0.\end{aligned}$$

Hence the CDF is well defined.

Proposition 5.2. The density function of PGPW class of distributions has an expanded linear representation of the form;

$$f(t) = n\lambda\gamma\theta\sum_{n=1}^{\infty} P(N=n)t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}. \tag{5.16}$$

Proof. By inputting $C'(\alpha) = \sum_{n=1}^{\infty} na_n\alpha^{n-1}$ into the PDF of the PGPW class of distributions, we have;

$$f(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times \sum_{n=1}^{\infty} na_n\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]^{n-1}}{C(\alpha)}$$

we simplify by the steps below to obtained the expanded form;

$$\begin{aligned}f(t) &= \lambda\gamma\theta\sum_{n=1}^{\infty}\frac{na_n}{C(\alpha)}t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}\alpha^{1+n-1}e^{[1-(1+\lambda t^{\gamma})^{\theta}]}e^{(n-1)(1-(1+\lambda t^{\gamma})^{\theta})}\\ &= \lambda\gamma\theta\sum_{n=1}^{\infty}\frac{na_n\alpha^{n}}{C(\alpha)}t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}\times e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}\end{aligned}$$

but $P(N=n) = \frac{a_n\alpha^n}{c(\alpha)}$

$$\begin{aligned}f(t) &= \lambda\gamma\theta\sum_{n=1}^{\infty} P(N=n)nt^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}\\ &= n\lambda\gamma\theta\sum_{n=1}^{\infty} P(N=n)t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}\end{aligned}$$

The above simplified representation of the PDF of the PGPW class of distributions helps to study its statistical properties such as moments, MGF, incomplete moments among others.

5.2.1 Sub families of the Power Series Generalised Power Weibull class of distributions

From the power series generalised power Weibull class of distributions (PGPW), four major sub-families of distributions can be obtained. These include; the generalised power geometric family of distributions (GPGD), the generalised power poisson family of distributions (GPPD, the generalised power binomial family of distributions (GPBD) and the generalised power logarithmic family of distributions (GPLD). These families are obtained by inputting the special cases of the zero truncated power series distributions in the PGPW class of distributions.

5.2.1.1 Generalized Power Geometric Family of Distributions

The geometric distribution truncated at zero is a distinct case of the power series distributions with $a_n = 1$, $C(\alpha) = \alpha(1-\alpha)^{-1}$ and $C'(\alpha) = (1-\alpha)^{-2}$. By inputting these functions into the PGPW class of distributions, we obtain the generalized power geometric family of distributions (GPGD) with CDF and PDF given as;

$$F(t) = \frac{1 - e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{1 - \alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}. \tag{5.17}$$

and

$$f(t) = \frac{(1-\alpha)\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda)^{\theta})}}{\left(1 - \alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{2}}. \tag{5.18}$$

$\alpha\epsilon(0,1)\text{or}(-\infty,1)$

Proof. Inputting, $a_n = 1$ and $C(\alpha) = \alpha(1-\alpha)^{-1}$ into the CDF of the PGPW class of distributions, the PGG family of distributions' CDF is defined as;

$$\begin{aligned}
F(t) &= 1-\left[\frac{\frac{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{\frac{\alpha}{1-\alpha}}\right] \\
&= 1-\left[\frac{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} \times \frac{1-\alpha}{\alpha}\right] \\
&= 1-\left[\frac{e^{[1-(1+\lambda t^{\gamma})^{\theta}]}(1-\alpha)}{1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}\right] \\
&= 1-\frac{e^{[1-(1+\lambda t^{\gamma})^{\theta}]}-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} \\
&= \frac{1-e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}
\end{aligned}$$

also for the PDF, we have;

$$\begin{aligned}
f(x) &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times \frac{1}{\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^2}}{\frac{\alpha}{1-\alpha}} \\
&= \alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times \frac{1}{\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^2} \times \frac{1-\alpha}{\alpha} \\
&= \frac{(1-\alpha)\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda)^{\theta})}}{\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^2}
\end{aligned}$$

The survival, hazard and reserved hazard functions of this family of distributions are given respectively as;

$$s(t) = \frac{e^{[1-(1+\lambda t^{\gamma})^{\theta}]}-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}{1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}. \tag{5.19}$$

$$h(t) = \frac{\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda)^{\theta})}}{e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)}. \tag{5.20}$$

$$R(t) = \frac{(1-\alpha)\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda)^{\theta})}}{\left(1-e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)}. \tag{5.21}$$

From the PGG family, the following distributions can be obtained;

The geometric Weibull distribution if $\theta = 1$ with CDF and PDF given respectively by;

$$F_{GW}(t) = \frac{1-e^{-\lambda t^{\gamma}}}{1-\alpha e^{-\lambda t^{\gamma}}}. \tag{5.22}$$

and

$$f_{GW}(t) = \frac{(1-\alpha)\lambda\gamma t^{\gamma-1}e^{-\lambda t^{\gamma}}}{(1-\alpha e^{-\lambda t^{\gamma}})^2}. \tag{5.23}$$

The geometric exponential distribution when $\gamma = 1$ and $\theta = 1$ with CDF and PDF defined as;

$$F_{GE}(t) = \frac{1-e^{-\lambda t}}{1-\alpha e^{-\lambda t}}. \tag{5.24}$$

and

$$f_{GE}(t) = \frac{(1-\alpha)\lambda e^{-\lambda t}}{(1-\alpha e^{-\lambda t})^2}. \tag{5.25}$$

Lastly the geometric NH distribution when $\gamma = 1$ with CDF and PDF given as;

$$F_{GNH}(t) = \frac{1-e^{(1-(1+\lambda t)^{\theta})}}{1-\alpha e^{(1-(1+\lambda t)^{\theta})}}. \tag{5.26}$$

and

$$f_{GNH}(t) = \frac{(1-\alpha)\lambda\theta(1-\lambda t)^{\theta-1}e^{(1-(1+\lambda t)^{\theta})}}{(1-\alpha e^{(1-(1+\lambda t)^{\theta})})^2}. \tag{5.27}$$

The plot of the PDF of the GPGD is displayed in Figures 5.1 and 5.2. The plots shows that, the PDF of this class of distribution can be decreasing, increasing, bathtub, unimodal and symmetric.

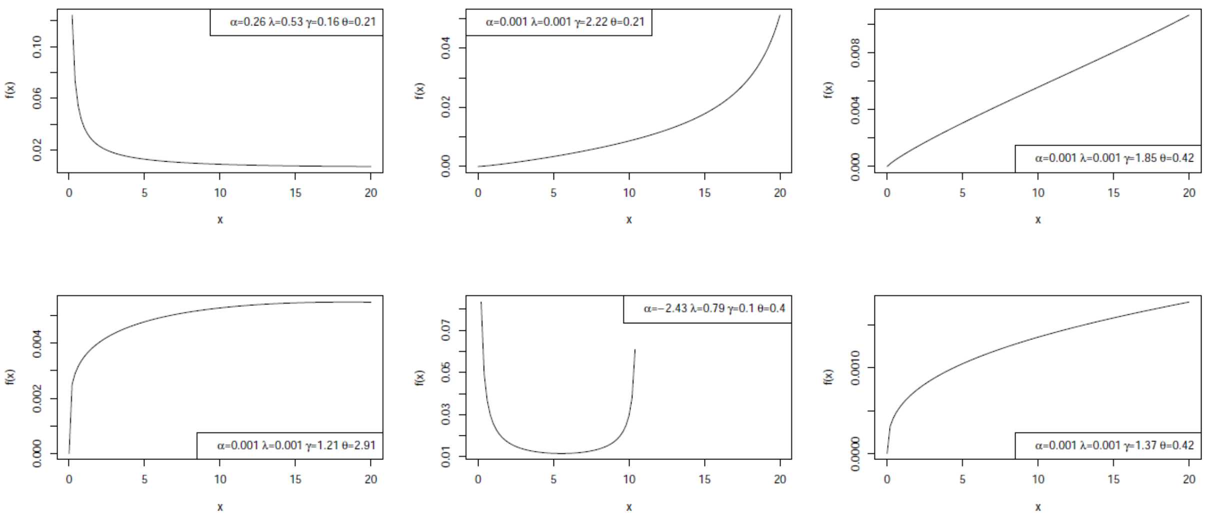

Figure 5.1: **PDF plot of the GPGD**

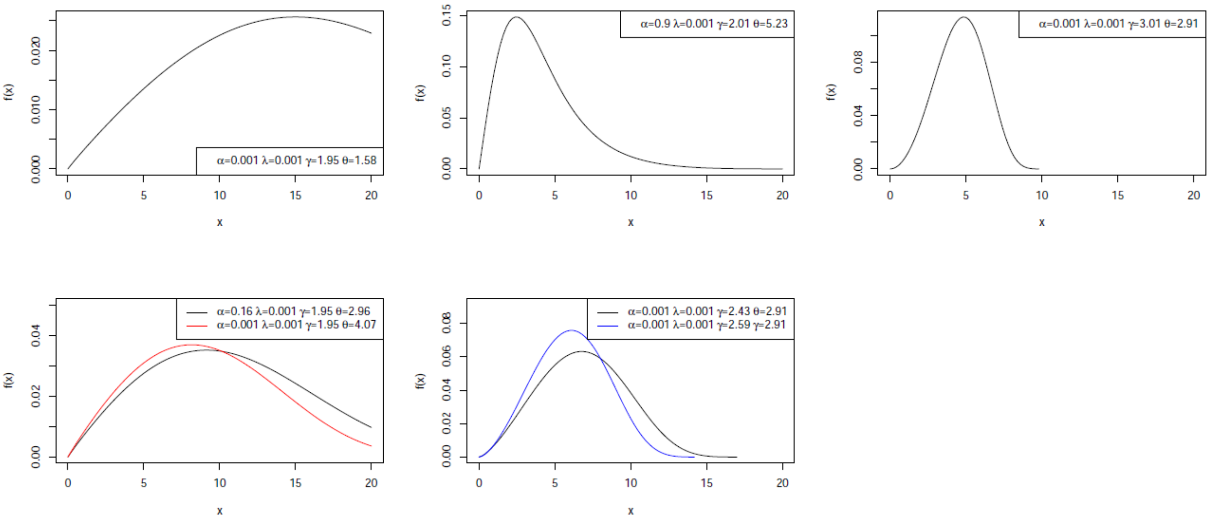

Figure 5.2: **PDF plot of the GPGD**

Also, the hazard plots of this class of distribution is displayed in Figures 5.3 and 5.4. It is seen that the hazard can be increasing, decreasing, bathtub and unimodal. This shows that the GPGD can model failure rate data which are both monotonic and non-monotonic shaped.

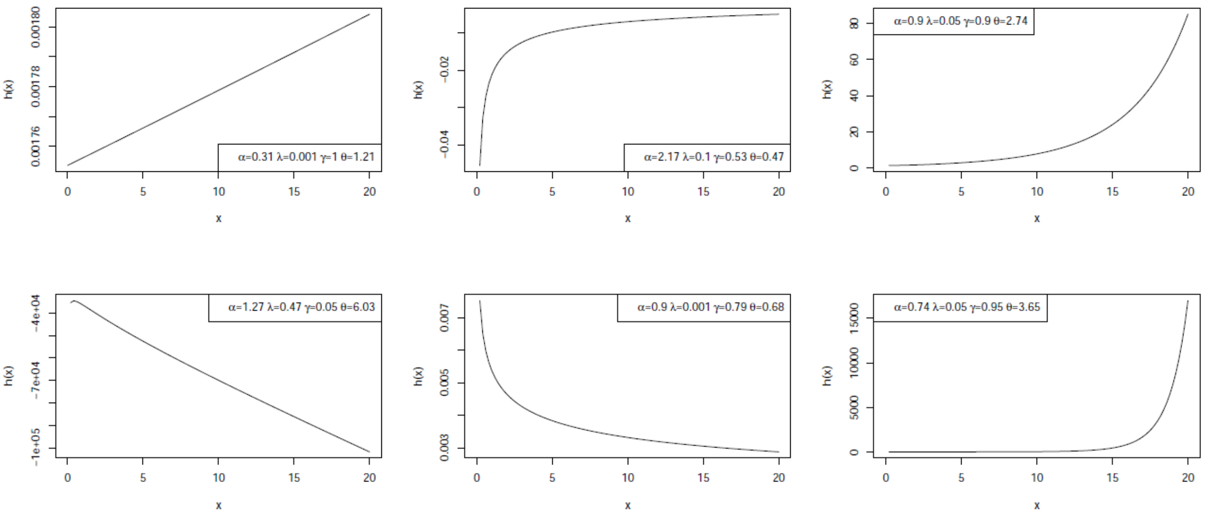

Figure 5.3: **Hazard plot of the GPGD**

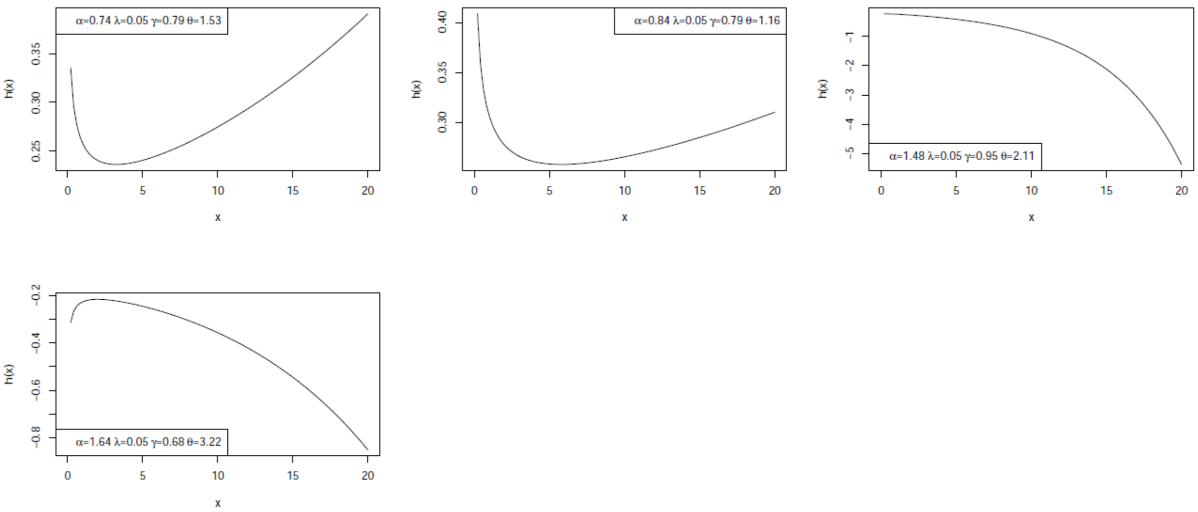

Figure 5.4: **Hazard plot of the GPGD**

5.2.1.2 Generalized Power Poisson Family of Distributions

The poison distribution (truncated at zero) is a special form of power series distribution with $a_n = \frac{1}{n!}$, $C(\alpha) = e^{\alpha} - 1$ and $C'(\alpha) = e^{\alpha}$. By inputting these functions in the PGPW class of distributions, we obtain the CDF and PDF of the generalized power Poisson family of distributions (GPPD) as;

$$F(t) = \frac{e^{\alpha} - e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha} - 1}. \tag{5.28}$$

and

$$f(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha} - 1}. \tag{5.29}$$

Proof. Considering $C(\alpha) = e^{\alpha} - 1$ in the PGPW class of distributions, we have the CDF of the GPP family as;

$$\begin{aligned} F(t) &= 1 - \frac{e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} - 1}{e^{\alpha} - 1} \\ &= \frac{e^{\alpha} - 1 - e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} + 1}{e^{\alpha} - 1} \\ &= \frac{e^{\alpha} - e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha} - 1} \end{aligned}$$

The survival, hazard and reversed hazard functions of the GPPD are given respectively as;

$$s(t) = \frac{e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} - 1}{e^{\alpha} - 1}. \tag{5.30}$$

$$h(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} - 1}. \tag{5.31}$$

$$R(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha} - e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}. \tag{5.32}$$

Three distributions can also be obtained from the GPP family of distributions if the parameters assumes a value of 1. These are;

The Poisson Weibull if $\theta = 1$. Its CDF and PDF are given respectively as;

$$F_{PW}(t) = \frac{e^{\alpha} - e^{\alpha e^{-\lambda t^{\gamma}}}}{e^{\alpha} - 1}. \tag{5.33}$$

and

$$f_{PW}(t) = \alpha\lambda\gamma t^{\gamma-1} e^{-\lambda t^{\gamma}} \times \frac{e^{\alpha e^{-\lambda t^{\gamma}}}}{e^{\alpha} - 1}. \tag{5.34}$$

The Poisson exponential if $\theta = 1$ and $\gamma = 1$ with CDF and PDF given respectively as;

$$F_{PE}(t) = \frac{e^{\alpha} - e^{\alpha e^{-\lambda t}}}{e^{\alpha} - 1}. \tag{5.35}$$

$$f_{PE}(t) = \alpha\lambda e^{-\lambda t} \times \frac{e^{\alpha e^{-\lambda t}}}{e^{\alpha} - 1}. \tag{5.36}$$

If $\gamma = 1$, we have Poisson NH distribution with CDF and PDF given as;

$$F_{PNH}(t) = \frac{e^{\alpha} - e^{\alpha e^{[1-(1+\lambda t)^{\theta}]}}}{e^{\alpha} - 1}. \tag{5.37}$$

$$f_{PNH}(t) = \alpha\lambda\theta(1+\lambda t)^{\theta-1} e^{[1-(1+\lambda t)^{\theta}]} \times \frac{e^{\alpha e^{[1-(1+\lambda t)^{\theta}]}}}{e^{\alpha} - 1}. \tag{5.38}$$

The plot of the PDF of the GPPD displayed in Figures 5.5 shows that, its PDF can be decreasing, increasing and unimodal (upside down bathtub).

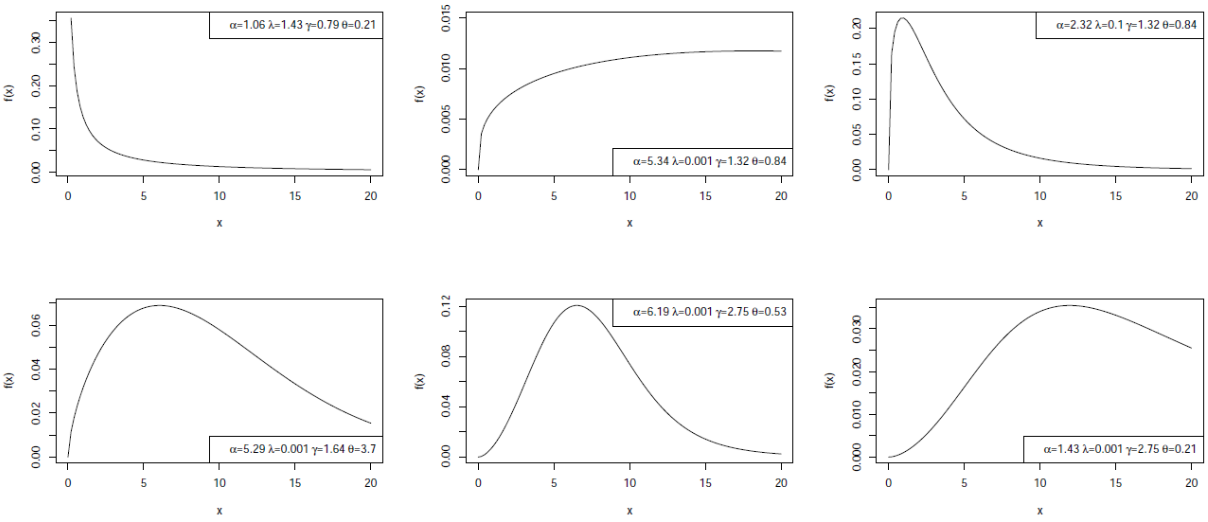

Figure 5.5: **PDF plot of the GPPD**

Also, the GPPD hazard rate is seen to be monotonically increasing, decreasing, bathtub, unimodal, modified bathtub and modified unimodal as shown in Figure 5.6 and Figure 4.7. This shows that the GPPD can model both monotonic and non-monotonic shaped failure rate.

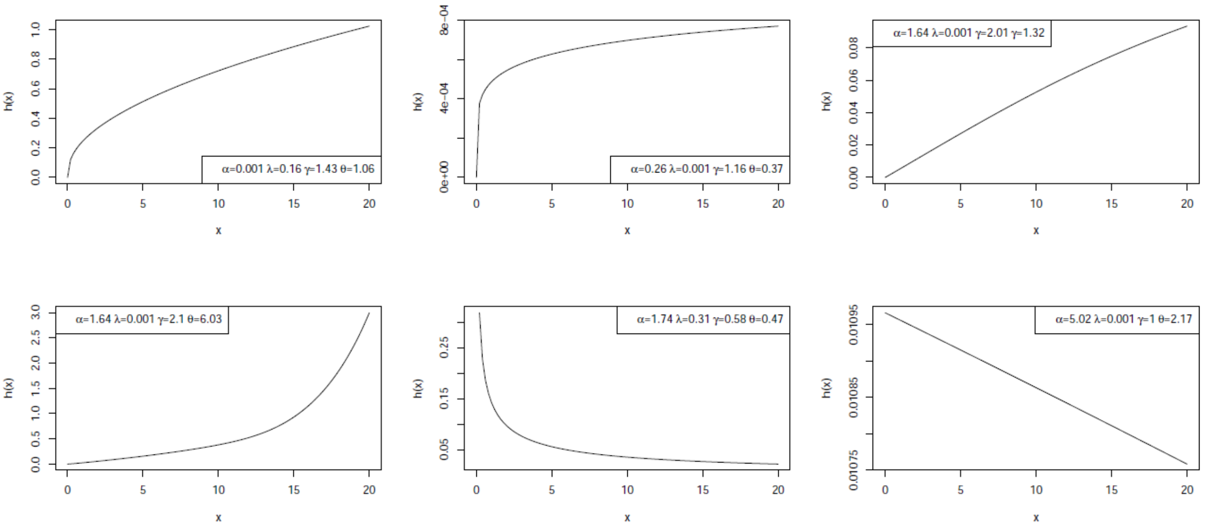

Figure 5.6: **Hazard rate plot of the GPPD**

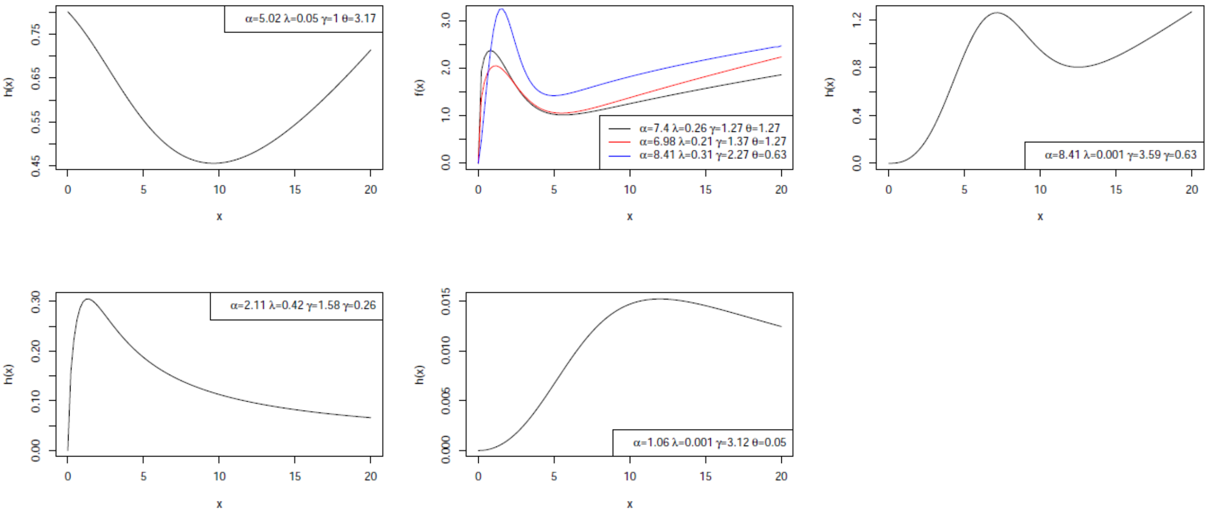

Figure 5.7: **Hazard rate plot of the GPPD**

5.2.1.3 Generalized Power Binomial Family of Distributions

The zero truncated binomial distribution is a special form of power series distributions with $a_n = \binom{m}{n}$, $C(\alpha) = (1+\alpha)^m - 1$ and $C'(\alpha) = \frac{m}{(1+\alpha)^{1-m}}$. By inputting these functions into the CDF and PDF of the PGPW class of distributions, we obtain the CDF and PDF of the generalized power binomial family of distributions (GPBD) respectively as;

$$F(t) = \frac{(1+\alpha)^m - (1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m}{(1+\alpha)^m - 1}. \tag{5.39}$$

and

$$f(t) = \frac{m\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}((1+\alpha)^m - 1)}. \tag{5.40}$$

Proof. If we consider $C(\alpha) = (1+\alpha)^m - 1$ in the PGPW distribution, we have;

$$\begin{aligned} F(t) &= 1 - \left[\frac{(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m - 1}{(1+\alpha)^m - 1}\right] \\ &= \frac{(1+\alpha)^m - 1 - (1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m + 1}{(1+\alpha)^m - 1} \\ &= \frac{(1+\alpha)^m - (1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m}{(1+\alpha)^m - 1} \end{aligned}$$

Also if we consider $C(\alpha) = (1+\alpha)^m - 1$ and $C'(\alpha) = \frac{m}{(1+\alpha)^{1-m}}$ in the PDF of the PGPW distribution, we have;

$$\begin{aligned} f(t) &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times m}{\frac{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}}{(1+\alpha)^m-1}} \\ &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times m}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}} \times \frac{1}{(1+\alpha)^m-1} \\ &= \frac{m\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}((1+\alpha)^m-1)} \end{aligned}$$

The survival, hazard and reversed hazard functions of the GPBD is given respectively as;

$$s(t) = \frac{(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m - 1}{(1+\alpha)^m - 1}. \tag{5.41}$$

$$h(t) = \frac{m\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}((1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m-1)}. \tag{5.42}$$

and

$$R(t) = \frac{m\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}((1+\alpha)^m-(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m)}. \tag{5.43}$$

From the GPB family, we can obtained three sub-distributions;

If $\theta = 1$, we have binomial Weibull distribution with CDF and PDF given respectively as;

$$F_{BW}(t) = \frac{(1+\alpha)^m - \left[1+\alpha e^{-\lambda t^{\gamma}}\right]^m}{(1+\alpha)^m - 1}. \tag{5.44}$$

and

$$f_{BW}(t) = \frac{m\alpha\lambda\gamma t^{\gamma-1}e^{-\lambda t^{\gamma}}}{((1+\alpha)^m-1)\left[(1+\alpha e^{-\lambda t^{\gamma}})\right]^{1-m}}. \tag{5.45}$$

If $\theta = 1$, $\gamma = 1$, we have binomial exponential distribution with CDF and PDF defined respectively as;

$$F_{BE}(t) = \frac{(1+\alpha)^m - \left[1+\alpha e^{-\lambda t}\right]^m}{(1+\alpha)^m - 1}. \tag{5.46}$$

and

$$f_{BE}(t) = \frac{m\alpha\lambda e^{-\lambda t}}{((1+\alpha)^m-1)\left[1+\alpha e^{-\lambda t}\right]^{1-m}}. \tag{5.47}$$

If $\gamma = 1$, we have the binomial NH with CDF and PDF defined as;

$$F_{BNH}(t) = \frac{(1+\alpha)^m - \left[1 + \alpha e^{(1-(1+\lambda t)^{\theta})}\right]^m}{(1+\alpha)^m - 1}. \tag{5.48}$$

and

$$f_{BNH}(t) = \frac{m\lambda\alpha\theta(1+\lambda t)^{\theta-1} e^{(1-(1+\lambda t)^{\theta})}}{((1+\alpha)^m - 1)\left(1 + \alpha e^{[1-(1+\lambda t)^{\theta}]}\right)^{1-m}}. \tag{5.49}$$

As displayed in Figures 5.8 and 5.9, the PDF of the GPBD can be increasing, decreasing, unimodal and positively skewed.

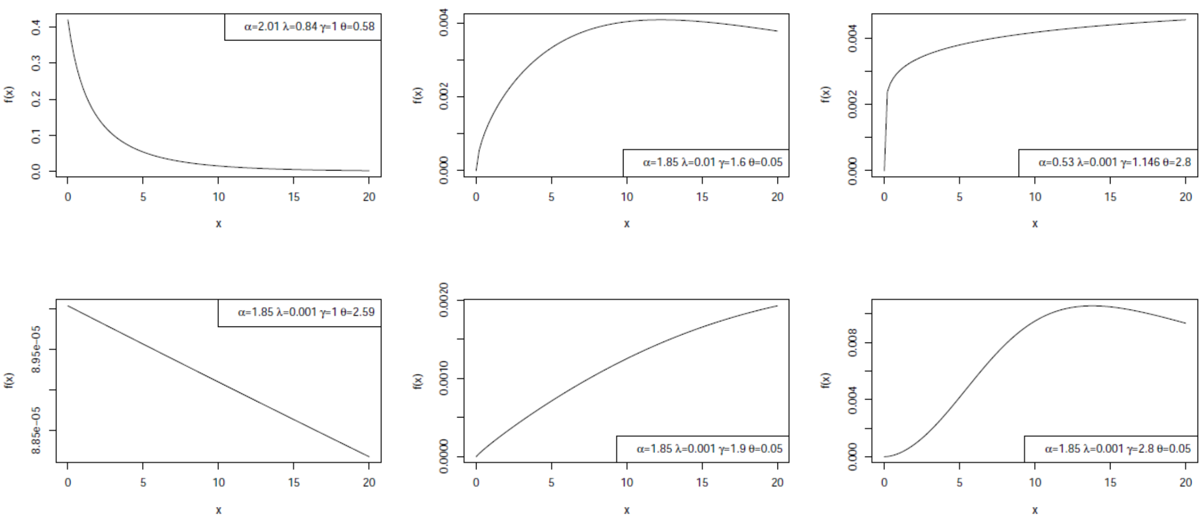

Figure 5.8: **PDF plot of the GPBD**

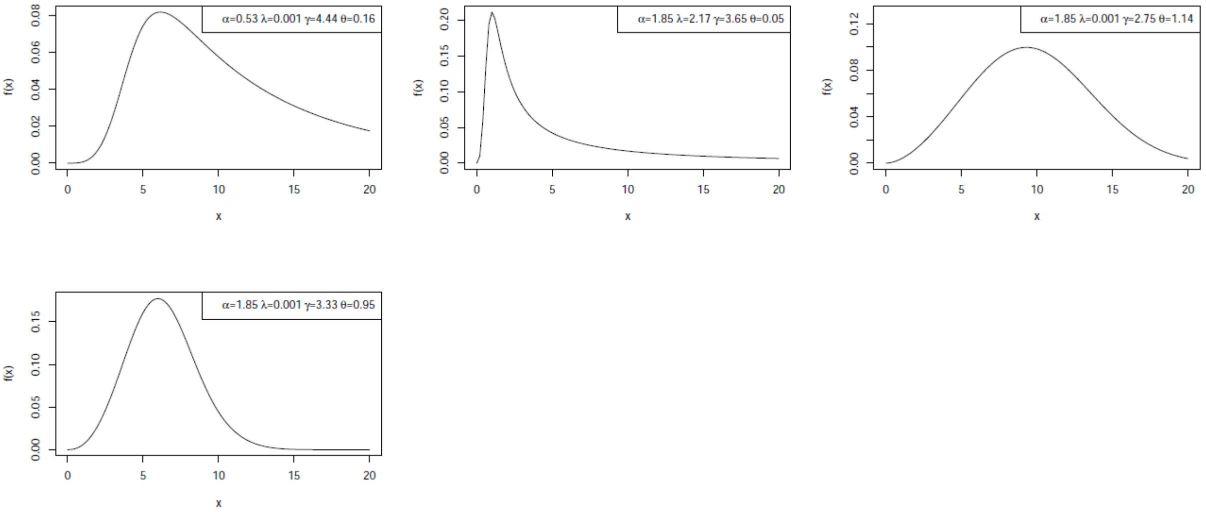

Figure 5.9: **PDF plot of the GPBD**

Also, its hazard function as shown in Figures 5.10 and 5.11 can be increasing, decreasing, bathtub and unimodal. This shows that the GPGD can model failure rate data which are both monotonic and non-monotonic shaped.

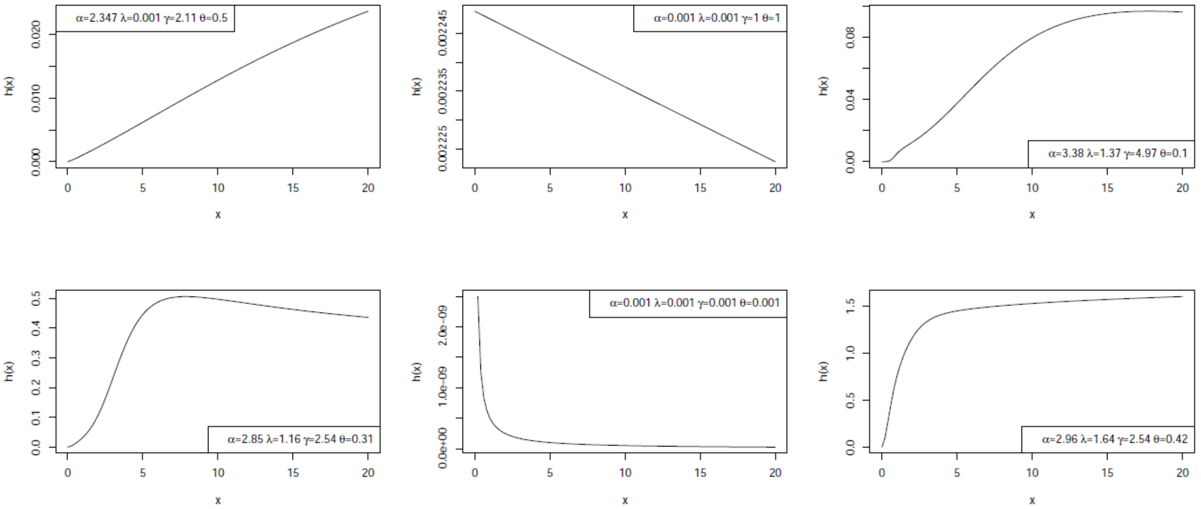

Figure 5.10: **Hazard rate plot of the GPBD**

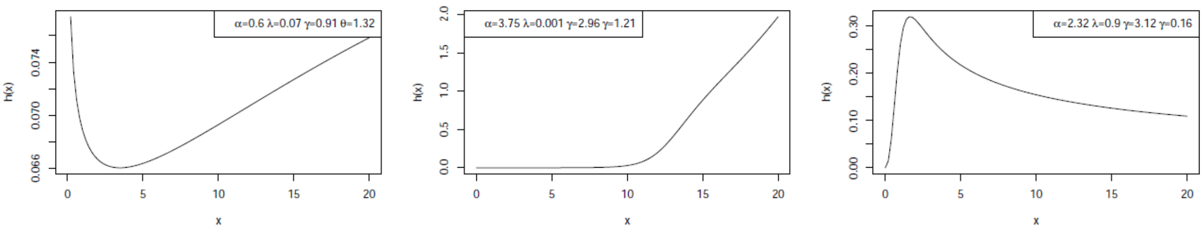

Figure 5.11: **Hazard rate plot of the GPBD**

5.2.1.4 Generalized Power Logarithmatic Family of Distributions

The zero truncated Logarithmatic distribution is also a special class of power series family with $a_n = \frac{1}{n}$, $C(\alpha) = -log(1-\alpha)$ and $C'(\alpha) = (1-\alpha)^{-1}$. From these quantities, the CDF and PDF of the generalized power logarithmatic family of distributions (PGLD) are given respectively as;

$$F(t) = 1 - \left[\frac{log\left(1 - \alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)}{log(1-\alpha)} \right]. \tag{5.50}$$

$$f(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}}{\left(\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]} - 1\right)(log(1-\alpha))}. \tag{5.51}$$

Proof. Considering $C(\alpha) = -log(1-\alpha)$ in the CDF of the PGPW class of distributions, we have;

$$\begin{aligned} F(t) &= 1 - \left[\frac{-\log\left(1 - \alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right)}{-\log(1-\alpha)} \right] \\ &= 1 - \left[\frac{\log\left(1 - \alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right)}{log(1-\alpha)} \right] \end{aligned}$$

Also considering $C(\alpha) = -log(1-\alpha)$ and $C'(\alpha) = (1-\alpha)^{-1}$ in the PDF of the PPGW distribution, we have;

$$\begin{aligned} f(t) &= -\frac{1}{log(1-\alpha)}\left[\frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})}\right] \\ &= -\left[\frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})log(1-\alpha)}\right] \\ &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}}{\left(\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}-1\right)(log(1-\alpha))}. \end{aligned}$$

The survival, hazard and reversed hazard functions of the GPLD are defined respectively as;

$$s(t) = \frac{\log\left(1-\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right)}{\log(1-\alpha)} \tag{5.52}$$

$$h(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}}{\left(\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}-1\right)\left(\log\left(1-\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right)\right)} \tag{5.53}$$

and

$$R(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}}{\left(\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}-1\right)\left(\log(1-\alpha)-\left(\log(1-\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right)\right)}. \tag{5.54}$$

For various values of the parameters, three sub distributions can also be obtained from the PGL family. If $\theta = 1$, we have logarithmic Weibull distribution with its CDF and PDF defined as;

$$F_{LW}(t) = 1 - \left[\frac{log(1-\alpha e^{-\lambda t^{\gamma}})}{log(1-\alpha)}\right]. \tag{5.55}$$

and

$$f_{LW}(t) = \frac{\alpha\lambda\theta t^{\gamma-1}e^{-\lambda t^{\gamma}}}{log(1-\alpha)(\alpha e^{-\lambda t^{\gamma}}-1)}. \tag{5.56}$$

If $\theta = 1$ and $\gamma = 1$, we have logarithmic exponential distribution having CDF and PDF defined as;

$$F_{LE}(t) = 1 - \left[\frac{log(1-\alpha e^{-\lambda t})}{log(1-\alpha)}\right]. \tag{5.57}$$

and

$$f_{LE}(t) = \frac{\alpha\lambda e^{-\lambda t}}{log(1-\alpha)(\alpha e^{-\lambda t}-1)}. \tag{5.58}$$

Lastly, if $\gamma = 1$, we have logarithmic NH distribution having CDF and PDF defined as;

$$F_{LNH}(t) = 1 - \left[\frac{log\left(1 - \alpha e^{\left[1-(1+\lambda t)^{\theta}\right]}\right)}{log(1-\alpha)}\right]. \tag{5.59}$$

and

$$f_{NH}(t) = \frac{\alpha\lambda\theta t(1+\lambda t)^{\theta-1} e^{\left[1-(1+\lambda t)^{\theta}\right]}}{log(1-\alpha)\left(\alpha e^{\left[1-(1+\lambda t)^{\theta}\right]} - 1\right)}. \tag{5.60}$$

5.2.2 Statistical Properties of the PPGW class of Distributions

This section discusses in details the distributional properties of the PGPW class of distributions. The properties considered are; the quantile function, ordinary (non-central) moments, moment generating function, order statistics, incomplete moment, mean deviation, median deviation, Lorenz and Bonferron curves, mean residual life, Stress-strength reliability and stochastic ordering property.

5.2.2.1 Quantile Function

The quantile function can serve as an alternative way of describing a probability distribution other than the probability density function, cumulative distribution function or characteristic function. It is the inverse of the cumulative distribution function. The quantile function can be used in both statistical application and Monte carlo methods. It can be used for generating random numbers from a given distribution.

Proposition 5.3. The quantile function of the PGPW class of distributions is;

$$Q_{F(p)} = \left[\frac{\left[1 - \log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta} - 1}{\lambda}\right]^{1/\gamma}. \tag{5.61}$$

where $C^{-1}(.)$ is the inverse of $C(.)$ and $p\epsilon[0,1]$.

Proof. By definition, the quantile function is defined as; $F(X_p) = P(x \leq x_p) = p$. Thus by setting, $Q_{F(p)} = t_p$ in equation 4.85, we have;

$$1 - \frac{C\left[\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right]}{C(\alpha)} = p$$

To make t the subject, we first make the exponent function the subject by the steps below;

$$\begin{aligned}\frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)} &= 1-p \\ C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right] &= (1-p).C(\alpha) \\ e^{[1-(1+\lambda t^{\gamma})^{\theta}]} &= \frac{C^{-1}(1-p).C(\alpha)}{\alpha}\end{aligned}$$

we take $\log_{10}$ on both sides and then make t the subject which gives the quantile function.

$$\begin{aligned}(1-(1+\lambda t^{\gamma})^{\theta}) &= \log\left[\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right] \\ (1+\lambda t^{\gamma})^{\theta} &= 1-\log\left[\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right] \\ (1+\lambda t^{\gamma}) &= \left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta} \\ \lambda t^{\gamma} &= \left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1 \\ t^{\gamma} &= \frac{\left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda} \\ t &= \left[\frac{\left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}\end{aligned}$$

$$\Rightarrow Q_{F(p)} = \left[\frac{\left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}.$$

Using the quantile function above, the median of the PGPW class of distributions evaluated at $p = 0.5$ is;

$$Q_{F(0.5)} = \left[\frac{\left[1-\log\left(\frac{C^{-1}(0.5).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}. \tag{5.62}$$

For the GPGD, $c^{-1}(\alpha) = \alpha(1+\alpha)^{-1}$ and $c(\alpha) = \alpha(1-\alpha)^{-1}$. Therefore, the GPGD quantile is given as;

$$Q_{GPGDF(p)} = \left[\frac{\left[1-\log\left(\frac{(1-p)}{1-\alpha p}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}. \tag{5.63}$$

For the GPPD, $c^{-1}(\alpha) = \log(1+\alpha)$ and $c(\alpha) = e^{\alpha}-1$. Hence its quantile function is given as;

$$Q_{GPPDF(p)} = \left[\frac{\left[1-\log\left(\frac{\log(1+((1-p)(e^{\alpha}-1))}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}. \tag{5.64}$$

For the GPBD, $c^{-1}(\alpha) = (1+\alpha)^{\frac{1}{m}}-1$ and $c(\alpha) = (1+\alpha)^{m}-1$. Hence its quantile function is given as;

$$Q_{GPBDF(p)} = \left[\frac{\left[1-\log\left(\frac{((((1-p)((1-\alpha)^{m}-1)))+1)^{\frac{1}{m}}}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}. \tag{5.65}$$

For the GPLD, $c^{-1}(\alpha) = 1-e^{-\alpha}$ and $c(\alpha) = -\log(1-\alpha)$. Hence its quantile function is given as;

$$Q_{GPLDF(p)} = \left[\frac{\left[1-\log\left(\frac{\left(1-e^{(1-p)\log(1-\alpha)}\right)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}. \tag{5.66}$$

5.2.3 Moments of the Power Series Generalised Power Weibull Family of Distributions

In statistical analysis, the moments of a random variable plays an vital in calculating various measures of variation (for instance; variance, standard deviation, mean deviation, coefficient of variation etc). It is used in the computation of the skewness and kurtosis of the distribution of the random variable.

Proposition 5.4. The r^{th} non-central moment of the PGPW class of distributions is given as;

$$U_r^1 = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n(-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]. \tag{5.67}$$

Where $\Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]$ is a complementary incomplete gamma function.

Proof. By definition, The r^{th} non-central moment of a random variable is given as;

$$\mu_r^{'} = \int_{-\infty}^{\infty} t^r f(t)dt.$$

for the PGPW class,

$$\mu_r^{'} = \int_0^{\infty} t^r \sum_{n=1}^{\infty} P(N=n)g_1(t)dt.$$

Substituting the linear expanded form of the PDF of the PGPW class of distributions, we have;

$$\begin{aligned}
\mu_r^{'} &= \int_0^{\infty} t^r \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n)t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}dt \\
&= \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n) \int_0^{\infty} t^r t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}dt
\end{aligned}$$

but $e^{n[1-(1+\lambda t^{\gamma})^{\theta}]} = e^n.e^{-n(1+\lambda t^{\gamma})^{\theta}}$, hence we obtain;

$$\mu_r^{'} = \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n)e^n \int_0^{\infty} t^r t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{-n(1+\lambda t^{\gamma})^{\theta}}dt$$

Applying Integration by substitution to simplify $\mu_r^{'}$ we follow the steps outline below;

Let

$$u = n(1+\lambda t^{\gamma})^{\theta}, t = \left(\frac{\left(\frac{u}{n}\right)^{\frac{1}{\theta}} - 1}{\lambda}\right)^{\frac{1}{\gamma}},$$

then

$$\left\{ \begin{array}{ll} \text{t} \longrightarrow 0, & u \rightarrow n \\ \text{t} \longrightarrow \infty, & y \rightarrow \infty \end{array} \right\}.$$

Also,

$$\frac{du}{dt} = n\lambda\gamma\theta t^{\gamma-1}(1+\lambda t\gamma)^{\theta-1},$$

$$dt = \frac{du}{n\lambda\gamma\theta t^{\gamma-1}(1+\lambda t\gamma)^{\theta-1}}$$

hence;

$$\begin{aligned}
\mu_r^{'} &= \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n)e^n \int_n^{\infty} t^r t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{-n(1+\lambda t^{\gamma})^{\theta}} \frac{du}{n\lambda\gamma\theta t^{\gamma-1}(1+\lambda t\gamma)^{\theta-1}} \\
&= \sum_{n=1}^{\infty} P(N=n)e^n \int_n^{\infty} \left(\frac{\left(\frac{u}{n}\right)^{\frac{1}{\theta}}-1}{\lambda}\right)^{\frac{r}{\gamma}} e^{-u}du \\
&= \sum_{n=1}^{\infty} P(N=n)e^n \lambda^{-\frac{r}{\gamma}} \int_n^{\infty} \left(\left(\frac{u}{n}\right)^{\frac{1}{\theta}}-1\right)^{\frac{r}{\gamma}} e^{-u}du.
\end{aligned}$$

Using the generalised from of binomial expansion; $(x+y)^i = \sum_{j=1}^{i}\binom{i}{j}x^{i-j}y^j, |x|>|y|$. Since$\left|\left[\left(\frac{u}{n}\right)^{\frac{1}{\theta}}\right]\right| < 1$,$x = \left(\frac{u}{n}\right)^{\frac{1}{\theta}}, y=-1$. Therefore we have;

$$\begin{aligned}
\mu_r^{'} &= \sum_{n=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n \int_n^{\infty} \sum_{j=1}^{\infty}(-1)^j \binom{\frac{r}{\gamma}}{j} \left(\frac{u}{n}\right)^{\left(\frac{r}{\gamma}-j\right)^{\frac{1}{\theta}}} e^{-u}du \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n(-1)^j \binom{\frac{r}{\gamma}}{j} \left(\frac{1}{n}\right)^{\left(\frac{r}{\gamma}-j\right)^{\frac{1}{\theta}}} \int_n^{\infty} u^{\left(\frac{r}{\gamma}-j\right)^{\frac{1}{\theta}}} e^{-u}du \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n(-1)^j \binom{\frac{r}{\gamma}}{j} \left(\frac{1}{n}\right)^{\left(\frac{r-\gamma}{\gamma\theta}\right)} \int_n^{\infty} u^{\left(\frac{r-\gamma j}{\gamma\theta}\right)} e^{-u}du \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n(-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma}{\gamma\theta}\right)-1+1} \int_n^{\infty} u^{\left(\frac{r-\gamma j}{\gamma\theta}\right)-1+1} e^{-u}du \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n(-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \int_n^{\infty} u^{\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)} e^{-u}du \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^n(-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]
\end{aligned}$$

The r^{th} non-central moments can be used to estimate the central moments μ_r and cumulants (K_r).

$$U_r = \sum_{k=0}^{r}(-1)^k \binom{r}{k} U_1^{1k} U_{r-k}^1 \tag{5.68}$$

$$K_r = U_r^1 = \sum_{k=1}^{r-1}\binom{k-1}{r-1} K_k U_{r-k}^1 \tag{5.69}$$

$K_1 = U_1^1$ gives the mean, $K_2 = U_2^1 - (U_{1^1})^2$ gives the variance and $K_3 = U_3^1 - 3U_2^1 + 2(U_1^1)^3$ Therefore the skewness value is defined as

$$S = \frac{K_3}{(K_2)^{\frac{3}{2}}} \tag{5.70}$$

and the Kurtosis value also defined as;

$$K = \frac{K_4}{(K_2)^2} \tag{5.71}$$

5.2.4 Moment Generating Function

The MGF are distinct functions used determine the moments of a random variable.

Proposition 5.5. The MGF of the PGPW class of distributions is given as;

$$M_t(z) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{\infty} \frac{Z^r}{r!} e^n (-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} P(N=n) \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]. \tag{5.72}$$

Proof. By definition MGF is given as;

$$\begin{aligned} M_t(z) &= E(e^{tz}) \\ &= \int_0^{\infty} e^{tz} f(t) dt \end{aligned}$$

Using Taylor series to expand we have

$$\begin{aligned} M_t(z) &= \int_0^{\infty} \sum_{r=0}^{\infty} \frac{Z^r t^r}{r!} f(t) dt \\ &= \sum_{r=0}^{\infty} \frac{Z^r}{r!} \int_0^{\infty} t^r f(t) dt \\ &= \sum_{r=0}^{\infty} \frac{Z^r}{r!} \mu_r' \end{aligned}$$

but $\mu_r^{'} = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n)e^{n}(-1)^{j}\binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]$

$$\begin{aligned} M_t(z) &= \sum_{n=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty} \frac{Z^r}{r!}\lambda^{\frac{-r}{\gamma}} P(N=n)e^{n}(-1)^{j}\binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right] \\ &= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{\infty} \frac{Z^r}{r!} e^{n}(-1)^{j}\binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} P(N=n)\Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]. \end{aligned}$$

5.2.5 Order Statistics

Let $X_1, X_n; X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be a random sample of size n, then the pdf of the p^{th} order statistic is given as;

$$f_{p:n}(t) = \frac{n!}{(n-p)!(p-1)!}\left[F(t)\right]^{p-1}\left[1-F(t)\right]^{n-p} f(t).$$

Assuming $X_1, X_n; X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ comes from the PGPW class of distributions, then;

$$f_{p:n}(t) = \frac{n!}{(n-p)!(p-1)!} f(t)\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{p-1}\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-p}.$$

Proposition 5.6. The PDF of the largest order statistics $(p = n)$ for the PGPW class of distributions is given as;

$$f_{p:n}(t) = n\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})}\left[\frac{C'\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}. \tag{5.73}$$

Proof. For the largest order statistics, $p = n$, hence;

$$\begin{aligned} f_{p=n}(t) &= \frac{n(n-1)!}{(n-n)!(n-1)!} f(t)\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{0} \\ &= nf(t)\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}. \end{aligned}$$

Inputting the PDF of the PGPW class of distributions, we have

$$f_{p=n}(t) = n\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\left[\frac{C'\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}.$$

Proposition 5.7. The PDF of the smallest order, ($p = 1$) of the PGPW class of distributions is given as;

$$f_{p:1}(t) = n\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\left[\frac{C'\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}. \tag{5.74}$$

Proof. For the smallest order statistic, $p = 1$, hence we have;

$$\begin{aligned} f_{p:1}(t) &= \frac{n(n-1)!}{(n-1)!(1-1)!}f(t)\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{1-1}\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1} \\ &= nf(t)\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}. \end{aligned}$$

Inputting the PDF of the PGPW class of distributions, we have;

$$f_{p:1}(t) = n\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\left[\frac{C'\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}$$

5.2.6 Incomplete Moment

Incomplete moment plays a vital role in computing the mean deviation, median deviation, inequality measures and mean residual life of the distribution of a random. Incomplete moments can also be used to describe the shape of a distribution of a random variable.

Proposition 5.8. The r^{th} incomplete moment of the PGPW class of distributions is given as;

$$\begin{aligned} M_r(y) = \sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{r}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{r}{\gamma}}{j}n^{-\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)} \\ \times\left[\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda y^{\gamma}\right)^{\theta}\right)\right]. \end{aligned} \tag{5.75}$$

Proof. By definition, the r^{th} incomplete moment is given as;

$$M_r(y) = \int_0^y t^r f(t)dt$$

Using the linear expanded form of the PDF of the PGPW class of distributions, the incomplete moment can be written as;

$$\begin{aligned} M_r(y) &= \int_0^y t^r \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n) t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{n(1-(1+\lambda t^{\gamma})^{\theta})} dt \\ &= n\lambda\gamma\theta e^n \sum_{n=1}^{\infty} P(N=n) \int_0^y t^r t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{-n(1+\lambda t^{\gamma})^{\theta}} dt \end{aligned}$$

we further simplify using integration by substitution as shown below.

Let

$$u = n(1+\lambda t^{\gamma})^{\theta}$$

then

$$\left\{ \begin{array}{ll} \text{t} \longrightarrow 0, & u \to n \\ \text{t} \longrightarrow y, & y \to n(1+\lambda y^{\gamma})^{\theta} \end{array} \right\}.$$

Also,

$$dt = \frac{du}{n\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}}.$$

Therefore the incomplete moment is given as;

$$\begin{aligned} M_r(y) &= e^n \sum_{n=1}^{\infty} P(N=n) \int_n^{n(1+\lambda y^{\gamma})^{\theta}} \left(\frac{(\frac{u}{n})^{\frac{1}{\theta}} - 1}{\lambda} \right)^{\frac{r}{\gamma}} e^{-u} du \\ &= e^n \lambda^{-\frac{r}{\gamma}} \sum_{n=1}^{\infty} P(N=n) \int_n^{n(1+\lambda y^{\gamma})^{\theta}} \left(\left(\frac{u}{n}\right)^{\frac{1}{\theta}} - 1 \right)^{\frac{r}{\gamma}} e^{-u} du. \end{aligned}$$

Further simplify using binomial expansion, we have;

$$\begin{aligned}
M_r(y) &= \sum_{n=1}^{\infty}\lambda^{-\frac{r}{\gamma}}e^nP(N=n)\int_n^{n(1+\lambda y^\gamma)^\theta}\sum_{j=1}^{i}(-1)^j\binom{\frac{r}{\gamma}}{j}\left(\frac{u}{n}\right)^{\left(\frac{r}{\gamma}-j\right)\frac{1}{\theta}}e^{-u}du\\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{r}{\gamma}}e^nP(N=n)(-1)^j\binom{\frac{r}{\gamma}}{j}\int_n^{n(1+\lambda y^\gamma)^\theta}\left(\frac{u}{n}\right)^{\left(\frac{r-\gamma j}{\gamma\theta}\right)-1+1}e^{-u}du\\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{r}{\gamma}}e^nP(N=n)(-1)^j\binom{\frac{r}{\gamma}}{j}n^{-\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)}\int_n^{n(1+\lambda y^\gamma)^\theta}u^{\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)}e^{-u}du\\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{r}{\gamma}}e^nP(N=n)(-1)^j\binom{\frac{r}{\gamma}}{j}n^{-\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)}\times\\
&\quad\left[\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda y^\gamma\right)^\theta\right)\right].
\end{aligned}$$

5.2.7 Mean and Median Deviation

Mean and median deviations measure the deviations from the mean and median of a random variable and can serve as methods of determining the extent of spread in a population.

Proposition 5.9. The mean deviation of the PGPW class of distributions is given by;

$$D(\mu)=2\mu F(\mu)-2\sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^nP(N=n)(-1)^j\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}$$

$$\times\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda\mu^\gamma\right)^\theta\right)\right]. \quad (5.76)$$

Proof. The mean deviation of a random variable is given as;

$$\begin{aligned}
D(\mu) &= \int_0^\infty|x-\mu|f(t)dt\\
&= 2\mu F(\mu)-2\int_0^\mu tf(t)dt
\end{aligned}$$

But $\int_0^\mu tf(t)dt=m_1(\mu)$ is the first incomplete moment $(r=1)$.

$$\begin{aligned}
M_1(\mu) &= \sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^nP(N=n)(-1)^j\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}\times\\
&\quad\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda\mu^\gamma\right)^\theta\right)\right].
\end{aligned}$$

Therefore, the mean deviation then becomes;

$$\begin{aligned} D(\mu) &= 2\mu F(\mu) - 2\sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)} \times \\ & \left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda\mu^{\gamma}\right)^{\theta}\right)\right]. \end{aligned}$$

Proposition 5.10. The median deviation of the PGPW class of distributions is given by;

$$D(M) = -\mu + 2\sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}$$

$$\times\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda m^{\gamma}\right)^{\theta}\right)\right]. \quad (5.77)$$

Proof. By definition the median deviation is given as;

$$\begin{aligned} D(M) &= -\mu + 2\int_{m}^{\infty} t f(t)dt \\ &= -\mu + 2[M_1(m)] \end{aligned}$$

Inputting $M_1(m)$, we get

$$\begin{aligned} D(M) &= -\mu + 2\sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)} \times \\ & \left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda m^{\gamma}\right)^{\theta}\right)\right] \end{aligned}$$

5.2.8 Residual and Mean Residual Life

Mean residual life (MRL) function at time y can represent the estimated added life span for a unit alive at time y. For an operating system, its residual life at time y is $T_y = T - y|T > y$ which has PDF given as;

$$f(t, y) = \frac{f(t)}{1 - F(y)}.$$

Proposition 45.11. The MRL of the T_y from the PGPW class of distribution is given as;

$$MRL = \frac{A\Gamma\left[\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right)\right] - B\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda\mu^{\gamma}\right)^{\theta}\right)\right]}{\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}} - y. \tag{5.78}$$

where

$$A = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-1}{\gamma}} P(N=n) e^{n} (-1)^{j} \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\theta\gamma}\right)}$$

and

$$B = \sum_{n=1}^{\infty}\sum_{j=1}^{i} \lambda^{-\frac{1}{\gamma}} e^{n} P(N=n)(-1)^{j} \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}.$$

Proof. The MRL $(t>0)$ is defined as;

$$\begin{aligned} MLR &= E\left(T-y|T>y\right) \\ &= \frac{\int_{y}^{\infty}(t-y)f(t)dt}{1-F(t)} \\ &= \frac{\mu_1^{'} - \int_0^y tf(t)dt}{1-F(t)} - y \end{aligned}$$

But $\int_0^y tf(t)dt = M_1(y)$ gives the fist incomplete moment and μ_1^1 gives the first non-central moment. Substituting these, the MRL is obtained.

5.2.9 Lorenz and Bonferron Curves

Loren and Bonferroni curves are used to measure the inequalities in the distribution of a random variable (for example income inequality). These curves are mostly applicable in reliability, medical, demographic, insurance and economic fields. For the PGPW class of distributions, the Loren curve is given as;

$$L(p) = \frac{1}{\mu}\sum_{n=1}^{\infty}\sum_{j=1}^{i} \lambda^{-\frac{1}{\gamma}} e^{n} P(N=n)(-1)^{j} \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}$$

$$\times\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda y^{\gamma}\right)^{\theta}\right)\right]. \tag{5.79}$$

Proof. By definition, the Lorenz curve is given as;

$$L(P) = \frac{1}{\mu}\int_0^y tf(t)dt.$$

But $\int_0^y tf(t)dt = M_1(y)$ is the first incomplete moment. Hence imputing $M_1(y)$ in $L(P)$, the Lorenz curve expression is obtained.

Also, the Bonferron curve is defined as;

$$B(P) = \frac{L(P)}{F(y)}.$$

Therefore the Bonferron curve for the PGPW class of distributions is given as;

$$B(p) = \frac{\frac{1}{\mu}\sum_{n,j=1}^{i\infty,i} \lambda^{-\frac{1}{\gamma}} e^n P(N=n)(-1)^j \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)} \left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda y^{\gamma}\right)^{\theta}\right)\right]}{1 - \frac{C\left[\alpha e^{(1-(1+\lambda y^{\gamma})^{\theta})}\right]}{C(\alpha)}} \tag{5.80}$$

5.2.10 Stochastic Ordering

This is used to compare two random variables to know which of them is larger or smaller. Stochastic ordering is an ordering mechanism in lifetime distribution. Assuming random variables $T_1 \sim PGPW(t,\alpha,\lambda,\gamma,\theta)$ and $T_2 \sim PGPW(t,\lambda,\gamma,\theta)$. Then T_1 is said to be greater than T_2 in likelihood ratio order if $\frac{f_{T_1}(t)}{f_{T_2}(t)}$ is an increasing function of T.

Proposition 4.12. Let $T_1 \sim PGPW(t,\alpha,\lambda,\gamma,\theta)$ and $T_2 \sim PGPW(t,\lambda,\gamma,\theta)$, then T_2 is greater than T_1 ($T_1 \leq_{lr} T_2$) for $\alpha > 0$.

Proof. For $T_1 \sim PGPW(t,\alpha,\lambda,\gamma,\theta)$ and $T_2 \sim PGPW(t,\lambda,\gamma,\theta)$,

$$\begin{aligned}\frac{f_{T_1}(t)}{f_{T_2}(t)} &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \frac{C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}}{\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})}} \\ &= \frac{\alpha C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}.\end{aligned}$$

$$\frac{d}{dt}\left[\frac{f_{T_1}(t)}{f_{T_2}(t)}\right] = -\alpha^2\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} \frac{C''\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}$$

since $\frac{d}{dt}\left[\frac{f_{T_1}(t)}{f_{T_2}(t)}\right] < 0$ for all $t > 0$, $\frac{d}{dt}\left[\frac{f_{T_1}(t)}{f_{T_2}(t)}\right]$ is a decreasing function for $\alpha > 0$.

5.3 Parameter Estimation

In this section, the unknown parameters of the PGPW class of distributions were estimated using the maximum likelihood estimation technique.

5.3.1 Maximum Likelihood Estimation

MLE finds the parameter estimates by determining the values of the parameters that maximize $L(\theta; X)$. Assuming $X = (X_1, X_2, ..., X_n)$ are measurement values of a random variable with density function $f(X;\theta)$, where θ is the parameter value from the distribution, then MLE finds the value of the model parameter θ, that maximizes $L(\theta; X)$. MLE estimators were obtained for the four sub-families of the PGPW class of distribution.
For the PGP class distribution, the likelihood function is given as;

$$L = n\log(\alpha\lambda\gamma\theta) + (\gamma-1)\sum_{i=1}^{n} log(t_i) + (\theta-1)\sum_{i=1}^{n} log(1+\lambda t_i^{\gamma}) + \sum_{i=1}^{n}(1-(1+\lambda t^{\gamma}))$$

$$\times\alpha\sum_{i=1}^{n} e^{1-(1+\lambda t^{\gamma})^{\theta}} - n\log(e^{\alpha}-1). \tag{5.81}$$

To obtain the MLE of the parameters, we maximises the score function by taking the first derivative of it. These are;

$$\frac{\partial l}{\partial \alpha} = ne^{(1-(1+\lambda t^{\gamma}))^{\theta}} - e^{\alpha}\log n + \log n\gamma\theta\lambda. \tag{5.82}$$

$$\frac{\partial l}{\partial \lambda} = n\log(\gamma\theta\alpha) - nt^{\gamma}\theta(1+\lambda t^{\theta})^{\theta-1} - nt^{\gamma}\alpha\theta((1+\lambda t^{\gamma})^{\theta-1})e^{1-(1+\lambda t^{\gamma})^{\theta}} + (\theta-1)\sum_{i=1}^{n}\log t_i. \tag{5.83}$$

$$\frac{\partial l}{\partial \gamma} = n\log\alpha\theta\lambda - n\theta\lambda t^{\gamma}(1+t^{\gamma})^{\theta-1}\log(t) - n\alpha\theta\lambda t^{\gamma}(1+\lambda t^{\gamma})^{\theta-1}e^{1-(1+\lambda t^{\gamma})^{\theta}}\log t + \sum_{i=1}^{n}\log(t)$$

$$\times(\theta-1)\sum_{i=1}^{n}\log\lambda\log(t)t^{\gamma}. \tag{5.84}$$

$$\frac{\partial l}{\partial \theta} = n\log\alpha\gamma\lambda - n(1+\lambda t^{\gamma})^{\theta} log(1+\lambda t^{\gamma}) - n\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\log(1+\lambda t^{\gamma})^{\theta} + \sum_{i=1}^{n}\log(1+\lambda t^{\gamma}). \tag{5.85}$$

For the GPL class of distributions, the likelihood function is given as;

$$L = n\log(\alpha\lambda\gamma\theta)+(\gamma-1)\sum_{i=1}^{n}\log(t_i)+(\gamma-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma})+\sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta})-n\log(\alpha)$$

$$-\sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta})+\log(1)-n\log(\log(1-\alpha)). \quad (5.86)$$

and the MLE of the parameters of this class are;

$$\frac{\partial l}{\partial \alpha} = -\log n+\log^2 n+\log n\gamma\theta\lambda. \quad (5.87)$$

$$\frac{\partial l}{\partial \lambda} = \log n\alpha\gamma\theta+(\theta-1)\sum_{i=1}^{n}t_1^{\gamma}. \quad (5.88)$$

$$\frac{\partial l}{\partial \gamma} = \log n\alpha\gamma\theta\lambda+\sum_{i=1}^{n}\log t+(\theta-1)\sum_{i=1}^{n}\log\lambda\log t_i+t^{\gamma}. \quad (5.89)$$

$$\frac{\partial l}{\partial \theta} = \log n\alpha\gamma\theta\lambda+\sum_{i=1}^{n}\log(1+\lambda t^{\gamma}). \quad (5.90)$$

For the GPG class of distribution, the likelihood function is given as;

$$L = n\log(1-\alpha)(\lambda\gamma\theta)+(\gamma-1)\sum_{i=1}^{n}\log(t_i)+(\theta-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma})+\sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta})-2$$

$$\times\sum_{i=1}^{n}(1-\alpha e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}). \quad (5.91)$$

The MLE estimators of this class are;

$$\frac{\partial l}{\partial \alpha} = 2ne^{(1-(1+\lambda t_i^{\gamma})^{\theta})}-\log n\gamma\theta\lambda. \quad (5.92)$$

$$\frac{\partial l}{\partial \lambda} = \log n(1-\alpha)\gamma\theta-nt^{\gamma}\theta((1+\lambda t^{\gamma})^{\theta-1})-2nt^{\gamma}\alpha\theta e^{(1-(1+\lambda t_i^{\gamma})^{\theta-1})}+(\theta-1)\sum_{i=1}^{n}\log t_i^{\gamma}. \quad (5.93)$$

$$\frac{\partial l}{\partial \gamma} = \log n(1-\alpha)\gamma\lambda-nt^{\gamma}\alpha\theta(1+\lambda t^{\gamma})^{\theta-1}\log(t)-2nt^{\gamma}\alpha\theta e^{(1-(1+\lambda t_i^{\gamma})^{\theta-1})}$$

$$+\sum_{i=1}^{n}\log t+(\theta-1)\sum_{i=1}^{n}\log\lambda\log(t)t^{\gamma}. \quad (5.94)$$

$$\frac{\partial l}{\partial \theta} = \log n(1-\alpha)\gamma\lambda - n(1+\lambda t^{\gamma})^{\theta}\log(1+\lambda t^{\gamma}) - 2nt^{\gamma}\alpha\theta e^{(1-(1+\lambda t_i^{\gamma})^{\theta-1})} + \sum_{i=1}^{n}(1+\lambda t_i^{\gamma}). \quad (5.95)$$

The PGB class has its likelihood function defined as;

$$L = n\log(m\alpha\lambda\gamma\theta) + (\gamma-1)\sum_{i=1}^{n}\log(t_i) + (\theta-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma}) + \sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta}) - n\log((1+\alpha)^m - 1)$$

$$-(1-m)\sum_{i=1}^{n}(1+\alpha e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}). \quad (5.96)$$

The MLE parameter estimates of this class are given as;

$$\frac{\partial l}{\partial \alpha} = -\log mn(1+\alpha)^{m-1} + \log mn\gamma\theta\lambda - (m-1)\sum_{i=1}^{n} e^{(1-(1+\lambda t_i^{\gamma})^{\theta})} \quad (5.97)$$

$$\frac{\partial l}{\partial \lambda} = -\log mn\alpha\gamma\theta + (\theta-1)\sum_{i=1}^{n}\log t_i^{\gamma} - \sum_{i=1}^{n}\theta t_i^{\gamma}(1+\lambda t_i^{\gamma})^{\theta-1} + (1-m)\sum_{i=1}^{n} e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}\log\alpha\theta t_i^{\gamma}(1+\lambda t_i^{\gamma})^{\theta-1}. \quad (5.98)$$

$$\frac{\partial l}{\partial \gamma} = \log mn\alpha\gamma\theta\lambda + \sum_{i=1}^{n}\log t_i^{\gamma} - (\gamma-1)\sum_{i=1}^{n}\log\lambda\log(t_i)t_i^{\gamma} - \sum_{i=1}^{n}\theta\lambda\log(t_i)t_i^{\gamma}(1+\lambda t_i^{\gamma})^{\theta-1}$$

$$-(1-m)\sum_{i=1}^{n} e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}\log\alpha\theta\lambda\log(t_i)t_i^{\gamma}(1+\lambda t_i^{\gamma})^{\theta-1}. \quad (5.99)$$

$$\frac{\partial l}{\partial \theta} = \log mn\alpha\gamma\lambda + \sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma}) - \sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma})(1+\lambda t_i^{\gamma})^{\theta} - (1-m)\sum_{i=1}^{n} e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}\log\alpha\log(1+\lambda t_i^{\gamma})$$

$$\times(1+\lambda t_i^{\gamma})^{\theta}. \quad (5.100)$$

5.4 Monte Carlo Simulation

Simulation analyses was conducted to assess the performance of the maximum likelihood estimators for the parameters of the sub-families of the PGPW distribution (thus the GPGD, GPDP, GPBD and GPLD). Three parameter value combinations of each distribution were specified. The quantile function of each distribution was then used to generate five different random samples of sizes, $n = 40, 80, 120, 160, 200$. These were then used to obtain the maximum likelihood estimates of the parameters of them. With a replication for N=1000 times, the average bias (ABias) and mean square error (MSE) were calculated for the estimators of the parameters of each distribution. For the GPBD,

$m = 5$ was used for the simulation. The results of the simulation analyses are shown in Tables 5.1 to 5.4. The results showed that, the maximum likelihood estimates of the parameters of each distribution converges to the true parameter value since the average bias of each parameter decrease as the sample size increases and the mean square errors also approaches zero as the sample size increases.

Table 5.1: **Monte Carlo Simulation Results for the Parameters of the GPGD**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.3	0.7	2.5	0.5	0.288	59.780	1.332	0.438	0.093	4.89657.300	5.0734	1.403
80	0.3	0.7	2.5	0.5	0.283	13.181	0.632	0.417	0.091	110324.500	0.965	0.870
120	0.3	0.7	2.5	0.5	0.281	0.861	0.447	0.412	0.089	2.297	0.369	0.921
160	0.3	0.7	2.5	0.5	0.189	0.724	0.350	0.348	0.088	4.000	0.221	0.677
200	0.3	0.7	2.5	0.5	0.119	0.635	0.304	0.327	0.080	0.767	0.164	0.569
40	0.3	0.4	2.8	0.3	0.295	6.678	1.568	0.128	0.103	75.756	8.628	0.035
80	0.3	0.4	2.8	0.3	0.292	0.456	0.702	0.097	0.100	0.922	1.054	0.031
120	0.3	0.4	2.8	0.3	0.287	0.346	0.511	0.079	0.097	0.543	0.581	0.024
160	0.3	0.4	2.8	0.3	0.275	0.275	0.420	0.074	0.090	0.133	0.353	0.060
200	0.3	0.4	2.8	0.3	0.275	0.274	0.372	0.061	0.092	0.120	0.262	0.007
40	0.2	0.1	2.6	0.5	0.240	0.070	1.409	0.462	0.075	0.010	6.882	1.326
80	0.2	0.1	2.6	0.5	0.249	0.053	0.649	0.459	0.071	0.005	0.883	1.337
120	0.2	0.1	2.6	0.5	0.241	0.051	0.448	0.403	0.072	0.004	0.396	0.846
160	0.2	0.1	2.6	0.5	0.238	0.049	0.346	0.387	0.069	0.004	0.214	0.808
200	0.2	0.1	2.6	0.5	0.234	0.049	0.295	0.342	0.063	0.049	0.211	0.342

Table 5.2: **Monte Carlo Simulation Results for the Parameters of the GPPD**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.3	0.7	2.5	0.5	0.579	0.693	0.625	0.648	0.377	2.854	0.859	5.462
80	0.3	0.7	2.5	0.5	0.554	0.525	0.436	0.330	0.354	2.502	0.400	0.454
120	0.3	0.7	2.5	0.5	0.535	0.334	0.345	0.248	0.340	0.208	0.189	0.231
160	0.3	0.7	2.5	0.5	0.329	0.145	0.129	0.072	0.329	0.145	0.129	0.072
200	0.3	0.7	2.5	0.5	0.318	0.142	0.129	0.061	0.324	0.110	0.111	0.047
40	0.4	0.4	2.7	0.3	0.482	0.281	0.913	0.273	0.265	0.706	2.756	0.604
80	0.4	0.4	2.7	0.3	0.458	0.164	0.535	0.134	0.249	0.046	0.599	0.062
120	0.4	0.4	2.7	0.3	0.454	0.146	0.413	0.089	0.245	0.033	0.282	0.020
160	0.4	0.4	2.7	0.3	0.451	0.128	0.338	0.069	0.244	0.024	0.195	0.014
200	0.4	0.4	2.7	0.3	0.446	0.118	0.294	0.055	0.238	0.022	0.143	0.006
40	1.4	0.4	1.7	0.6	0.556	0.567	0.409	0.676	0.397	4.927	0.452	2.851
80	1.4	0.4	1.7	0.6	0.541	0.397	0.305	0.386	0.369	0.514	0.164	0.684
120	1.4	0.4	1.7	0.6	0.494	0.312	0.236	0.268	0.351	0.251	0.092	0.407
160	1.4	0.4	1.7	0.6	0.487	0.284	0.197	0.193	0.335	0.177	0.066	0.086
200	1.4	0.4	1.7	0.6	0.319	0.272	0.176	0.159	0.332	0.144	0.051	0.062

Table 5.3: **Monte Carlo Simulation Results for the Parameters of the GPLD**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.3	0.7	2.5	0.5	0.358	0.980	0.651	0.284	0.158	111.428	1.039	0.451
80	0.3	0.7	2.5	0.5	0.358	0.341	0.392	0.237	0.161	0.428	0.279	0.234
120	0.3	0.7	2.5	0.5	0.355	0.310	0.309	0.186	0.156	0.176	0.165	0.132
160	0.3	0.7	2.5	0.5	0.346	0.298	0.255	0.148	0.152	0.138	0.115	0.056
200	0.3	0.7	2.5	0.5	0.344	0.297	0.229	0.136	0.150	0.143	0.090	0.044
40	0.4	0.7	2.5	0.5	0.322	1.385	0.666	0.355	0.126	144.378	1.078	0.760
80	0.4	0.7	2.5	0.5	0.301	0.370	0.413	0.219	0.110	0.481	0.316	0.125
120	0.4	0.7	2.5	0.5	0.279	0.329	0.353	0.198	0.100	0.325	0.245	0.148
160	0.4	0.7	2.5	0.5	0.278	0.279	0.248	0.153	0.100	0.108	0.099	0.064
200	0.4	0.7	2.5	0.5	0.280	0.271	0.250	0.145	0.099	0.104	0.097	0.056
40	0.9	0.3	1.5	0.6	0.302	3.859	0.427	0.484	0.169	876.771	0.669	1.396
80	0.9	0.3	1.5	0.6	0.265	0.373	0.216	0.252	0.132	0.288	0.075	0.189
120	0.9	0.3	1.5	0.6	0.254	0.388	0.170	0.225	0.118	0.257	0.043	0.144
160	0.9	0.3	1.5	0.6	0.251	0.377	0.147	0.161	0.113	0.252	0.034	0.045
200	0.9	0.3	1.5	0.6	0.208	0.303	0.144	0.156	0.096	0.185	0.030	0.052

Table 5.4: **Monte Carlo Simulation Results for the Parameters of the GPBD**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.5	0.7	2.5	0.5	0.368	0.696	40.963	0.454	0.164	0.484	1362.307	0.207
80	0.5	0.7	2.5	0.5	0.343	0.688	40.781	0.454	0.148	0.482	1343.548	0.204
120	0.5	0.7	2.5	0.5	0.331	0.685	39.344	0.441	0.140	0.479	1109.620	0.191
160	0.5	0.7	2.5	0.5	0.310	0.680	38.716	0.438	0.128	0.475	2300.550	0.189
200	0.5	0.7	2.5	0.5	0.303	0.501	48.699	0.420	0.123	0.429	1070.184	0.182
40	0.6	0.8	2.1	0.4	0.326	0.788	32.353	0.377	0.124	0.632	2559.651	0.142
80	0.6	0.8	2.1	0.4	0.297	0.785	31.972	0.362	0.109	0.621	2100.949	0.132
120	0.6	0.8	2.1	0.4	0.282	0.777	31.235	0.360	0.100	0.619	1966.925	0.125
160	0.6	0.8	2.1	0.4	0.268	0.777	30.872	0.358	0.091	0.537	1943.259	0.123
200	0.6	0.8	2.1	0.4	0.248	0.769	30.738	0.355	0.081	0.537	1729.232	0.117
40	0.3	0.9	1.5	0.7	0.492	0.899	64.255	0.679	0.291	0.797	4453.962	0.469
80	0.3	0.9	1.5	0.7	0.420	0.893	61.790	0.658	0.228	0.773	4453.173	0.468
120	0.3	0.9	1.5	0.7	0.395	0.885	61.333	0.651	0.202	0.701	4450.426	0.465
160	0.3	0.9	1.5	0.7	0.373	0.878	60.735	0.584	0.183	0.694	4436.201	0.462
200	0.3	0.9	1.5	0.7	0.337	0.872	60.154	0.553	0.152	0.606	43134.009	0.436

5.5 Applications

The derived GPGD, GPPD, GPBD (with $m = 5$) and the GPLD were applied to two sets of data (failure times data of air conditioning system of aircraft and the service time of 63 aircraft). The performance of these distributions in terms to providing good parametric fit to the two data sets were compared using the Kolmogorov Smirnov (KS) statistic, Cram$\acute{e}$r-Von Mises statistic (W*), Anderson- Darling statistic (A), log-likelihood and model selection criteria such as the AIC, AICc and BIC.

5.5.1 Application I: Failure times of air conditioning system of an aircraft

The first application uses 30 observations from the failure times of air conditioning system of an aircraft. This dataa is displayed in Table 5.3 in Appendix A.

Table 5.5 gives the descriptive statistics for the failure times data for the air conditioning system of an aircraft. From the results, the data set is positively skewed and platykurtic in nature since the skewness value is positive and the kurtosis value less than three. This implies that, the distribution of this data set is less peaked as compared to the normal distribution and majority of the data points are clustered at the lower side of the distribution with a long tail to the right.

Table 5.5: **Des. Stats. of the Failure times of air conditioning system of an aircraft**

Statistic	Mean	St.Dev	CV	Median	Kurtosis	Skewness
Value	59.600	71.900	120.610	22.000	2.570	1.780

The TTT transformed plot of the failure times of air conditioning system of an aircraft as shown in Figure 5.12 is first convex in shape, followed by a concave shape which indicate that the hazard function of the this data set is bathtub shaped.

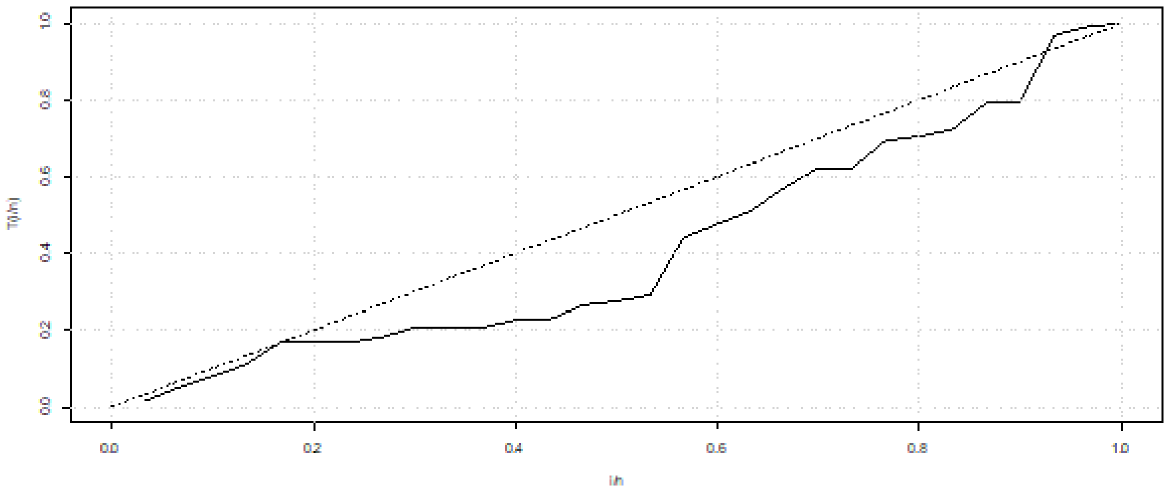

Figure 5.12: **TTT plot of failure times of the air conditioning system of an aircraft**

The detailed maximum likelihood parameter estimates for the four fitted families of distributions for the failure times of air conditioning system of an aircraft are shown in Table 5.6. By using the estimated standard errors and p-values for the four distributions, it is seen that all the parameters of the GPBD, GPGD and the GPLD are all significant at 5 percent significance level since their standard errors are less than half of their parameter estimates and their p-values are also less than 0.05. For the GPPD, all the parameters were significant at the 0.05 significance level with the exception of the parameter θ.

Table 5.6: **Maximum Likelihood Parameter Estimates, SE and p-values of failure times of air conditioning system of an aircraft**

Distribution	Par. Ests. and Std. errors			
GPBD	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$
	1.336	11.132	17.890	0.017
	(0.413)	(0.002)	(0.003)	(0.001)
	0.001	0.000	0.000	0.000
GPGD	-5.833	200.002	100.505	0.003
	(1.223×10^{-8})	(6.034×10^{-12})	(5.730×10^{-9})	(1.845×10^{-4})
	0.000	0.000	0.000	0.000
GPPD	2.812	0.007	0.960	0.960
	(1.032)	(0.008)	(0.255)	(2.096)
	(0.006)	(0.042)	(0.000)	$(0.560 **)$
GPLD	-39.157	99.891	105.586	0.003
	(1.202×10^{-9})	(1.066×10^{-11})	(5.285×10^{-9})	(1.857×10^{-4})
	(0.000)	(0.000)	(0.000)	(0.000)

Table 5.7 presents the likelihood, information criteria and goodness-of-fit measures for the fitted distributions for the failure times of air conditioning system of an aircraft. Among the four fitted families of distributions, the GPGD family has the largest log-likelihood value with the smallest Kolmogorov Smirnov (KS), Anderson-Darling (AD), Cramér-Von Mises (CVM), AIC, AICc, and BIC statistic values. These indicates that, the GPG family of distributions provides a better fit to the failure times of air conditioning system of an aircraft as compared to the other fitted distributions.

Table 5.7: **Goodness-of-fit and Information Criteria of failure times of air conditioning system of an aircraft**

Dist.	LL	$-2\log L$	AIC	AICc	BIC	CVM	AD	KS(p-value)
GPBD	-151.190	303.386	312.386	313.986	319.392	0.074	0.433	0.117(0.808)
GPGD	-151.170	302.348	310.348	311.948	315.953	0.075	0.471	0.118(0.798)
GPPD	-151.710	303.428	311.428	313.028	317.033	0.097	0.523	0.139(0.611)
GPLD	-151.990	303.984	311.984	313.584	317.589	0.075	0.517	0.183(0.268)

Figure 5.13 gives the plot of the empirical CDF and the CDFs of the GPGD, GPPD, GPBD and the GPLD for the failure times of air conditioning system of an aircraft. From the figure, the GPGD, GPPD and the GPLD distributions provides a better fit to the data than the GPBD.

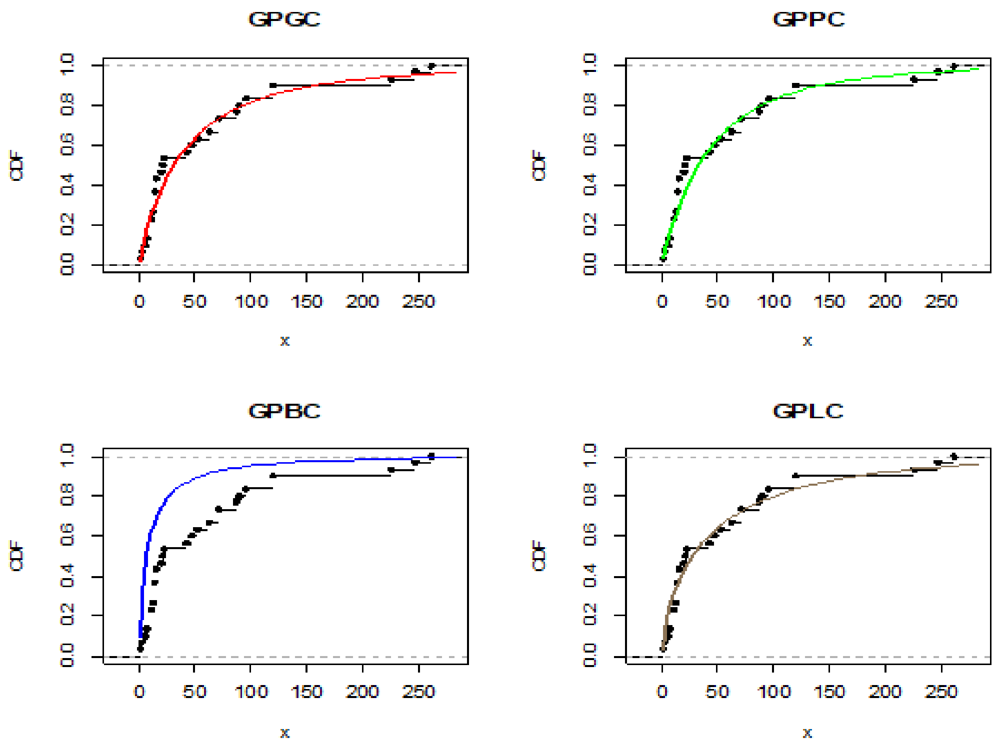

Figure 5.13: **Empirical CDF and CDF plots of failure times of air conditioning system of an aircraft**

5.5.2 Application II: Failure data on service times of 63 aircraft

The second applications of the four family of distributions used failure Data on service times of 63 aircraft. This failure rate data is given in Appendix A Table 6.4.

Table 5.8 gives the descriptive statistics for the failure data on service times of 63 aircraft. It is seen that the data set is positively skewed and platikurtic in nature since the skewness value is positive and the kurtosis value is less than three.

Table 5.8: **Descriptive Statistics of failure data on service times of 63 aircraft**

Statistic	Mean	St.Dev	CV	Median	Kurtosis	Skewness
Value	2.091	1.243	59.380	2.065	-0.170	0.450

The TTT transform plot of the service times of 63 aircraft as shown in Figure 5.14 indicated that, the data set has an unimodal (upside down bathtub) failure rate.

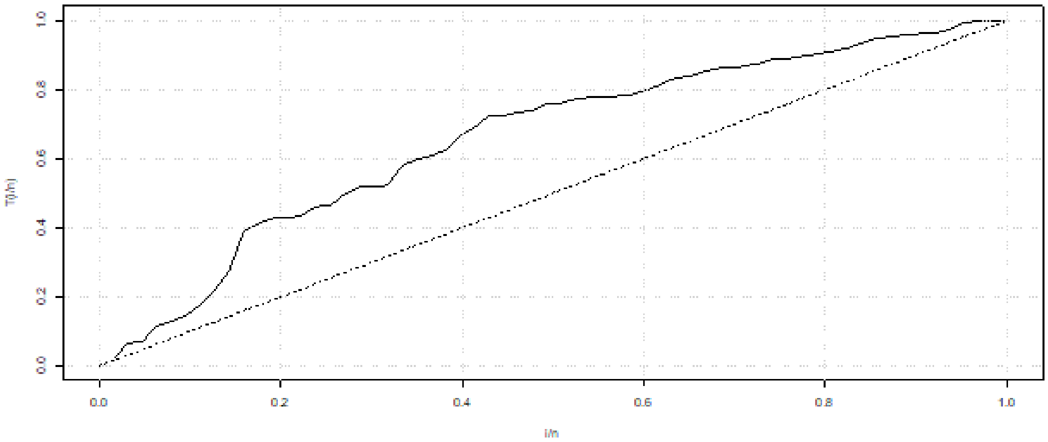

Figure 5.14: **TTT plot of service times of 63 aircraft**

The maximum likelihood parameter estimates, standard errors and p-values of the GPBD, GPGD, GPPD and the GPLD are presented in Table 5.9. Using the standard errors of the parameters, all the parameters estimates of the GPBD, GPGD and the GPLD are significant at 5 percent significant level. However, for the GPPD, λ, γ, θ are significant whiles α is not.

Table 5.9: **MLE Parameter Estimates, SE and p-values of the service times of 63 aircraft**

Distribution	Parameter Estimates			
GPBD	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$
	7.209	0.009	1.595	2.871
	(0.001)	(0.002)	(0.165)	(0.0010)
	0.000	0.000	0.000	0.000
GPGD	−65.228	197.833	1.910	0.253
	(0.002)	(0.001)	(0.279)	(0.016)
	0.000	0.000	0.000	0.000
GPPD	1.742	0.048	0.955	8.759
	(1.825)	(0.026)	(0.312)	(0.012)
	(0.340)	(0.065)	(0.002)	(0.000)
GPLD	−199.196	155.418	2.962	0.209
	(0.000)	(0.001)	(0.364)	(0.0126)
	(0.000)	(0.000)	(0.000)	(0.000)

The likelihood, goodness-of-fit and information criteria for the fitted distributions are presented in Table 5.10. The GPPD provides a better fit among the four fitted distributions since it has the highest log-likelihood and the minimum AIC, AICc, BIC, KS, AD, CVM and $-2\log L$ values.

Table 5.10: **Goodness-of-fit and Information Criteria of Kevlar 49/epoxy data**

Dist.	LL	$-2\log L$	AIC	AICc	BIC	W^*	A^*	K-S(p-value)
GPBD	−100.010	200.019	210.019	210.709	220.735	0.098	0.597	0.107(0.439)
GPGD	−100.690	201.388	209.385	210.075	217.958	0.101	0.620	0.085(0.717)
GPPD	−98.020	196.039	204.039	204.729	212.611	0.033	0.225	0.065(0.940)
GPLD	−104.380	208.769	290.656	291.346	299.228	0.045	0.286	0.462(0.268)

The plot of the empirical CDF and the CDF of the GPBD, GPPD, GPGD and GPLD are shown in Figure 5.15. From the plots, the GPGD and the GPPD provides a better fit as compared to the other distributions considered.

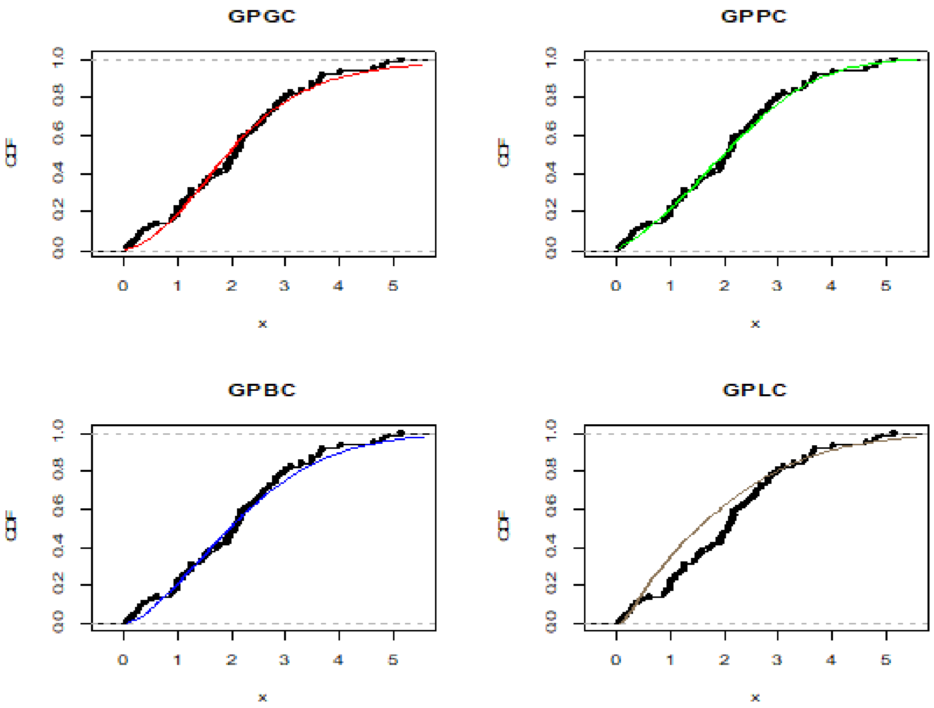

Figure 5.15: **Empirical CDF and CDF plots of service times of 63 aircraft**

CHAPTER 6

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

This chapter presents the summary, conclusion made from the study and recommendations for future works.

6.2 Summary

In this study, two new distributions for modelling lifetime data from series connected systems were presented. This made use of the concept of compounding. These distributions were proposed as alternatives for modelling lifetime data, although they are applicable in other fields. Firstly, the NHGPW distribution was developed for modelling lifetime data from systems connected in series by continuous-continuous compounding the NH and GPW distributions. It is obtained modelling the minimum between the NH and the GPW distributions. Some statistical properties such as quantiles, moments, moment generation function and order statistics were derived for the NHGPW distribution. The maximum likelihood estimators were derived for the parameters of this distribution. The analysis revealed that, the PDF of the NHGPW distribution can be monotonically decreasing, increasing, bathtub, unimodal, modified bathtub and symmetric. The hazard function of the NHGPW distribution can also be constant, monotonically increasing, decreasing, bathtub, unimodal and modified bathtub. This implies that, the NHGPW distribution can adequately model both monotonic and non-monotonic shaped failure rate data. The NHGPW distribution is very flexible because it contains several well known distributions as sub-distributions. Monte Carlo Simulation performed on this distribution (with 1000 replication) showed that, the MLE of the NHGPW distribution converges to the true parameter value since the mean square errors decay to zero and the biases of each parameter also decrease as the sample size increases. To demonstrate the flexibility of the NHGPW distribution, the distribution was applied on two lifetime sets. The parameter estimates of the NHPGW distribution for the two data set were significant at 5% significant level. Also, the NHGPW distribution provides a better fit to the Kevlar 49/epoxy data set and the Aircraft Windshield failure rate data as compared the competitive distributions since it has the highest log-likelihood value with the smallest Kolmogorov Smirnov (K-S), Anderson-Darling (A*), Crame′r-Von Mises (W*) statistics, smallest AIC, AICc, and BIC values. This can also be seen from the plot of the empirical CDF and the CDFs for both data set since its CDF approximates the empirical CDF. The likelihood ratio (LR) test performed for both data set showed that, the fit of the NHGPW is significantly different from its sub-distributions.

Secondly, the PGPW class of distributions was also developed. This was obtained by discrete-continuous compounding the power series family and the GPW distribution. Various statistical properties such as quantile function, moments, moment generation function, order statistics, stochastic ordering, incomplete moments, mean and median deviations, mean residual life and inequality curves were derived for the PGPW class of distributions. Four special family of distributions were derived from the PGPW class of distributions; thus the generalised power geometric (GPG) family, generalised power poison (GPP) family, generalised power binomial (GPB)family and the generalised power logarithmic (GPL) family. The statistical properties of these families of distributions were also presented. The parameter estimates of these family of distributions were also derived using maximum likelihood estimation. The shape of the PDFs of these four sub-families of distributions showed that, their PDFs can be monotonically increasing, decreasing, bathtub, symmetric and unimodal. Also, their hazard functions can be monotonically increasing, monotonically decreasing, bathtub, unimodal, modified bathtub and modified unimodal based on the parameter values. Hence,these distributions can model both monotone and non-monotone failure rate lifetime data. Monte Carlo simulation performed to assessed the maximum likelihood estimators of these sub-family of distribution showed that, their MLE estimators are consistent since their mean square error and average bias approaches zero as the sample size increase. To demonstrate the flexibility of the family of distributions, two lifetime data sets of lifetime were used. Application of the these families of distribution to failure data from air conditioning system of an aircraft indicates that, the GPG distribution provides a better fit as compared to the other fitted distributions. The GPP family also provides a better fit among the four fitted distributions for failure rate data of the service times of aircraft.

6.3 Conclusions

In this study, the Nadarajah Haghighi generalised power Weibull (NHGPW) distribution and the PGPW class of distributions were developed based on the concept of compounding. From the analysis, we conclude that, the cumulative distribution and probability distribution functions of developed NHGPW distribution were well defined and meet all necessary condition of a probability distribution. The plot of the PDF and hazard functions of the NHGPW distribution indicated that this distribution can adequately model both monotonic and non-monotonic failure rate data set since its PDF can be decreasing, increasing, bathtub, unimodal, modified bathtub and symmetric and its hazard function can also be constant, monotonically increasing, decreasing, bathtub, unimodal and modified bathtub.

The NHGPW distribution is also very flexible as compared to existing distributions since it contains several well known distributions as sub-distributions hence absorbed the desirable properties of these sub-distributions. Maximum likelihood estimators of the parameters of this distribution were presented. Monte Carlo simulation analysis on the maximum likelihood estimators showed that, the estimators of the NHGPW distribution were consistent since the converges to the true parameter value as the sample size increases. Based on the goodness of fit statistics, model selection criteria, the NHGPW distribution provided a better fit to the Kevlar 49/epoxy data set as compared the competitive distributions for systems connected in series. Also, the parameter estimates of the distribution were all significant at 5% significant level. The NHGPW distribution also provided a better as compared to the competitive distributions to the Aircraft Windshield failure rate data set. This can as well be seen from the plot of the empirical CDF and the CDFs for both data set since its CDF approximates the empirical CDF for both data set.

By discrete-continuous compounding the zero truncated power series family and the GPW distribution, the PGPW class of distributions were also developed. The PGPW class of distributions contains the generalised power geometric (GPG), generalised power poisson (GPP), generalised power binomial (GPB) and the generalised power logarithmic (GPL) as sub-families of distribution. As presented, the four sub-family of distributions of the PGPW class of distributions can adequately model both monotonic and non-monotonic lifetime data set since their PDFs and hazard functions exhibit various shapes such as monotonically increasing, decreasing, bathtub, unimodal among others. From the Monte Carlo simulation analysis, the estimators of each sub-family of distribution were consistent estimators since their mean square error and average bias approaches zero as the sample size increase. Application of the four family of distributions to failure time data of air conditioning system of an aircraft showed that, the GPG distribution provides a better fit among them whiles the GPP family also provided a better fit for failure rate data of the service times of aircraft.

6.4 Recommendations

- The NHGPW and PGPW distributions are applicable only to series connected systems where the least number of events is considered. It is recommended that, alternative distributions can be developed for modelling lifetime data set from parallel connected components where the maximum number of events is considered.
- The NHGPW and PGPW distributions considered a system with two subsystems connected in series. However, In lifetime data analysis, failure rate data from more than one system with sub-system connected in series might be encountered. It is therefore recommended that, a generalisation of the developed distributions can be made to accommodate n possible systems with n subsystems connected in series.
- In many life data applications, it possible to encounter situations where the failure rate of a system is affected by a number of covariates. It is therefore recommended that, further extensions of the NHGPW and PGPW distributions can be made to

include possible causal/independent variables.

- All applications done on the developed distributions used uncensored data set. Nonetheless, censored samples may arise in different fields of studies where the survival times of all the subjects are not exact and known. For example, patients lost to follow-up in a medical research, manufactured product not failing beyond the duration of a study etc. Hence, further studies should consider the use of censored data in demonstrating the applications of the developed distributions.

References

Aarset, M. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1):106–108.

Abdus, S., Muhammad, N. K., Cordeiro, M. G., Ibrahim, E., and Rodrigo, R. P. (2019). The beta exponentiated nadarajah-haghighi distribution: Theory, regression model and application.

Ahmed, Z. A., Cordeiro, G. M., Haithan, M. Y., Abdus, S., and Ortega, E. M. M. (2015). The marshall olkin additive weibull distribution. ams classifications. *AMS classifications*, pages 60E05, 62E15.

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Translation on Automatic Control*, AC-19:716–723.

Alamatsaz, M. and Shams, H. (2016). Generalized linear failure rate power series distribution. *Communications in Statistics - Theory and Methods*, 45(8):2204–2227.

Ali, A. J., Rasool, R., and Kundu, D. (2015). On bivariate generalized exponential-power series class of distributions. *AMS Subject Classification*, pages 62E15, 62H10.

Anderson, T. and Darling, D. (1954). A test of goodness of fit. *Journal of American Statistics Association*, 49:765–769.

Arthur, J. L., Cordeiro, M. G., and Ortega, E. M. (2014). On the additive weibull distribution. *Communications in Statistics-Theory and Methods*, 43:2066–2080.

Bagdonavicius, V. and Nikulin, M. (2002). Accelerated life models: modeling and statistical analysis. *Chapman and Hall/CRC, Boca Raton.*

Baitshephi, M., Oluyede, B. O., Fgbamigbe, A. F., and Makubate, B. K. S. (2019). A new class of generalized weibull-g family of distributions:theory, properties and applications. *International Journal of Statistics and Probability*, 8(1):73–93.

Barlow, R. E. and Doksum, K. A. (1972). Isotonic tests for convex orderings. *In Proceedings of the 6th Berkeley Symposium*, 1:293–323.

Bourguignon, M., J., L., Maria, L., and Abraão, D. C. N. (2015). A new generalized gamma distribution with applications. *American Journal of Mathematical and Management Sciences*.

Chahkandi, M. and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics and Data Analysis*, 53:4433–4440.

Cicero, D. R. B., Morad, A., and Cordeiro, M. G. (2000). The beta nadarajah-haghighi distribution. *Hacettepe Journal of Mathematics and Statistics*, 47(5):1302–1320.

Cohen, A. C. (1973). The reflected weibull distribution. *Technometrics*, 15(4):867–873.

Cordeiro, G. M., Ortega, E., and Nadarajah, S. (2010). The kumaraswamy weibull distribution with application to failure data. *J. Franklin Institute*, 347:1399–1429.

Cordeiro, G. M., Ortega, E. M., and Lemonte, A. J. (2014). The exponential-weibull lifetime distribution. *Journal of Statistics Comput. Simulation*, 84:2592–2606.

Cordeiro, G. M., Ortega, E. M. M., and Silva, G. O. (2012). Kumaraswamy modified weibull distribution: theory and applications. *J. of stat. comput. Simulation*, 84(71):1387–1411.

Cordeiro, M. G., Abdus, S., Muhammad, N. K., P., S. B., and Ortega, M. M. E. (2017). The transmuted generalised modified weibull distribution. *University of Nis, Faculty of Sciences and Mathematics*, 31(5):1395–1412.

Cordeiro, M. G., Haithan, M. Y., Ahmed, Z. A., and Selen, C. (2018). The lindley-weibull distributions; properties and application. *Annals of the Brazilian academic of science*, 90(3):2579–2598.

Cordeiro, M. G., William, D. A., Dora, M. S., and Lozano, S. (2015). The gamma modified weibull distribution. *Chilean Journal of Statistics*, 6(1):37–48.

Cramer, H. (1928). On the composition of elementary errors. second paper: Statistical aapplications. *Skand. Aktuartidskr*, 11:141–80.

Ehab, M. A. and Hisham, M. A. (2018). Estimation of the generalized power weibull distribution parameters using progressive censoring schemes international. *Journal of Probability and Statistics*, 7(2):51–61.

Eisa, M. and Mitra, S. (2012). Exponentiated weibull power series distributions and its applications. *arXiv preprint*, arXiv: 1212.5613.

Felipe, R. S., Gusmao, M. D., Ortega, M. M., and Cordeiro, M. (2005). The generalized inverse weibull distribution. 52(3):591–619.

Fernando, A. P.-R., Renata, R. G., Cordeiro, G. M., and Pedro, R. D. M. (2017). The exponentiated power generalized weibull: Properties and applications. *Annals of the Brazilian Academy of Sciences*, 90(3):2553–2577.

Fernando, A. P.-R., Renata, R. G., and Cordeiro, M. G. (2019). The nh lindley distribution. *Annals of the Brazilian academy of science*, 91(1).

Fernando, A. P.-R., Renata, R. G., Cordeiro, M. G., and Marinho (2018). The exponential power generalised weibull: properties and applications. *Annals of Brazilian Academy of science*, 90(30):2553–2577.

Fify, A., Nofal, A., Yousof, Z., H.M, G., and N.S, B. (2015). The transmuted weibull lomax distribution: properties and application. *Pakistan Journal of Statistics and Operation Research*, 11:135–152.

Flores, J., Borges, P., Cancho, V. G., and Louzada, F. (2013). The complementary exponential power series distribution.

Foya, S., Oluyede, B., A., F., and B., M. (2017). The gamma log-logistic weibull distribution model: properties and application. *Electron J. Applied Stat. Anal*, 10(1):206–241.

Gayan, W. L. and P., M. (2015). The lindley power series class of distributions: Model, properties and applications. *Journal of Computations & Modelling*, 5 (3):35–80.

Gupta, R. C. (1974). Modified power series distribution and some of its applications. *The Indian Journal of Statistics, Series B*, 36(3):288–298.

Gupta, R. D. and Kundu, D. (1982). Probability generating function of the modified power series distribution with application. *Mathematische operationfor schung and statistic, series statistics*, 13:99–103.

Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and weibull distributions. *Biom. J.*, 43:117–130.

Gupta, R. D. and Kundu, D. (2007). Generalized exponential distribution: Existing results and some recent development. *J. Stat. Plan. Inf.*, 137:3537–3547.

Gupta, S. R. and Singh, J. S. (1981). Soil respiration in tropical grassland. *Soil Biology and Biochemistry*, 13:261–268.

Hilary, I. O., Pelumi, E. O., Abiodun, A. O., and Sheila, A. B. (2018). Marshall-olkin-nadarajah-haghighi distribution: Ordinary differential equations. In *Proceedings of the World Congress on Engineering and Computer Science*, I, pages 23–25.

Hurvich, C. M. and Tsai, C. L. (1989). Regression and time series model selection in small sample. *Biometrika*, 76(2):297–307.

Ibrahim, E., Francisco, L., and Daniele, C. T. G. (2018). A new lifetime model: The kumaraswamy extension exponential distribution. *Biostatistics and Bioinformatics*, pages 2578–0247.

Jose, F., B, P., G.C, V., and Francisco, L. (2013). The complementary exponential power series distribution. *Braz. J. Probab. Stat*, 27(4):565–584.

Keller, A. Z., Kamath, A. R. R., Peacock, S. T., and Selman, A. C. (1984). Proposed methodology for assessment of reliability, maintainability and availability of medical equipment. *Reliability Engineering*, 9(3):153–174.

Kies, J. A. (1958). The strength of glass. *Naval Research Lab. Report*, 5093.

Kumar, A. and Consul, P. (1979). Negative moments of a modified power series distribution and bias of the maximum likelihood estimator. *Commun. Statist.-Theor. Meth*, 8(2):151–166.

Lai, C. D. (2013). Constructions and applications of lifetime distributions. *Applied Stochastic Models in Business and Industry*, 29:127–140.

Law, A. M. and Kelton, W. D. (2000). *Simulation Modeling and Analysis. 3rd ed.* Series in industrial engineering and management science, 3rd ed. New York.

Lemonte, A. J. (2013). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. *Computational Statistics and Data Analysis*, 62:149–170.

Lima, S. R. L. (2015). The half-normal generalized family and kumaraswamy nadarajah-haghighi distribution. Master's thesis, Universidade Federal de Pernambuco.

Mahmoud, A. S. (2018). Estimation and prediction for the nadarajah-haghighi distribution based on record values. *Pak. J. Statist.*, 34(1):77–90.

Mahmoud, A. S. and Abdullah, M. B. (2016). The kumaraswamy-generalized power weibull distribution. *Mathematical Theory and Modeling*, 6(2):ISSN 2224–5804.

Mahmoudi, E. and Jafari, A. A. (2011). Generalized exponential-power series distributions. *Computational Statistics and Data Analysis*, 56:4047–4066.

Mahmoudi, E. and Jafari, A. A. (2012). Generalized exponential power series distributions.

Marcelo, B., Maria, d. C. S. L., Jeremias, L., Abraão, D. C. N., Luis, G. B. P., and Cordeiro, G. M. (2015). A new generalized gamma distribution with applications. *American Journal of Mathematical and Management Sciences*, 34(4):309–342.

Marinho, P. R. D. (2016). *Some new families of continuous distributions.* PhD thesis, Universidade Federal de Pernambuco.

Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika*, 84:641–652.

Mdlongwa, P., Oluyede, B., Amey, A., Fagdamyb, A., and Mukabate, B. (2019). Kumaraswamy log-loistic weibull distribution: model, theory and applications to lifetime and survival data. *Heliyon 5. Doi.101016/s.heliyon.2019.e01144.*

Morais, A. L. W. and Barreto, S. (2011). A compound class of weibull and power series distributions. *Computational Statistics and Data Analysis*, 55:1410–1425.

Mudholkar, G. S. and Kollia, G. D. (1994). Generalized weibull family: a structural analysis. *Communications in statistics-theory and methods*, 23(4):1149–1171.

Nadarajah, S. and Haghighi, F. (2011). An extension of the exponential distribution. *Journal of theoretical and applied statistics*, 45(6):543–558.

Nadarajah, S. and R, R. (2017). Bivariate weibull-power series class of distributions.

Nasiru, S. (2016). Serial weibull rayleigh distribution; theory and application. *International journal of computing science and mathematics*, 7(3):239–244.

Nasiru, S. (2018). *A new generalization of transformed-transformer family of distributions.* PhD thesis, Pan African University institute for basic sciences, technology and innovation.

Nikulin, M. and Haghighi, F. (2006). A chi-squared test for the generalized power weibull family for the head-and-neck cancer censored data. *Journal of Mathematical Sciences*, 133:1333 –1341.

Nikulin, M. and Haghighi, F. (2009). On the power generalized weibull family. *Metron*, 67:75–86.

Oluyede, B. O., Warahena-Liyanage, and M., P. (2016). A new compound class of log-logistics weibull-poisson distribution: model, properties and applications. *J. of stat. comput. Simulation*, 86(7):1363–1391.

Said, H. A. (2015). Generalized extended weibull power series family of distributions. *American review of mathematics and statistics*, 3(2):53–68.

Sarhan, A. M. and Zaindin, M. (2009). Modified weibull distribution. *Applied Sciences*, 11:123–136.

Schwarz, G. E. (1978). Estimating the dimension of a model. *Annals of Statistics*, 6:461–464.

Shanmugam, R. (2001). Product moment generating function technique and its applications to modified power series distributions. *Communications in statistics-theory and methods*, 30(11):2385–2392.

Shoukri, M. M. and Consul, P. C. (1982). Bivariate modified power series distribution some properties, estimation and applications. *Biometrical Journal*, 24(8):787–799.

Silva, R., Bourguignon, M. C., Dias, R. C. B., and Cordeiro, G. (2013). The compound class of extended weibull power series distributions. *Computational Statistics and Data Analysis*, 58:352–367.

Silva, R. B. and Cordeiro, G. M. (2015). The burr xii power series distributions: A new compounding family. *Brazilian Journal of Probability and Statistics*, 29(3):565–589.

Stacy, E. W. (1962). A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, 33(3):1187–1192.

Stenio, R. L. (2015). *The Half-Normal Generalized Family and Kumaraswamy Nadarajah-Haghighi Distribution.* PhD thesis, Federal University Of Pernambuco, Center of Natural and Exact Sciences, Department of Statistics.

Sujata, R. and Rajash, M. B. (1988). Bathtub distribution. *Communications in Statistics-Theory and methods*, 17:2597–2621.

Tahir, M. H., Cordeiro, G. M., Sajid, A., Sanku, D., and Aroosa, M. (2018). The inverted nadarajah-haghighi distribution: estimation methods and applications. *Journal of Statistical Computation and Simulation*, pages 1563–5163.

Tripathi, R. C., Gupta, P. L., and Gupta, R. C. (1986). Incomplete moments of modified power series distributions with applications. *Journal Communications in Statistics-Theory and Methods*, 15(3).

Vedo, V. T., Nascimento, A., Miranda, W., Lima, M., Pinho, G. L., and Cordeiro, M. (2016). Some computational and theoretical aspects of the exponentiated generalized nadarajah-haghighi distribution. *Chilean Journal of Statistics*, Month 200x, arXiv:1610.08867.

Weibull, W. (1939). A statistical theory of the strength of materials. *Ing. VetenskapsAkad. Handt*, 151.

Xie, M. and Lai, C. D. (1996). Reliability analysis using an additive weibull model with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, 52(1):87–93.

Zhang, T. and Xie, M. (2007). Failure data analysis with extended weibull distribution. *Communication Statistical Simulation and Computation*, 36(3):579–592.

Appendix A

Table 6.1: **Failure Times Data of Kevlar 49/epoxy Strands at 90 Stress Level**

0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09
0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.29	0.34
0.35	0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.63	0.65	0.67
0.68	0.72	0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.85	0.90	0.92	0.95
0.99	1.00	1.01	1.02	1.03	1.05	1.10	1.10	1.15	1.18	1.2	1.29	1.31
1.33	1.34	1.4	1.43	1.45	1.5	1.51	1.53	1.54	1.54	1.55	1.58	1.60
1.63	1.64	1.80	1.80	1.81	2.02	2.14	2.17	2.33	3.03	3.34	4.20	4.69
7.89												

Table 6.2: **Failure times data of 84 Aircraft Windshield**

4.167	1.281	3.00	4.035	2.3	3.344	4.602	1.757	2.324	2.265	3.578	0.943	4.121
1.303	2.089	2.632	2.135	2.962	2.688	2.902	0.557	1.911	1.568	3.595	1.07	4.255
1.899	2.61	3.478	1.248	2.01	1.194	1.505	2.154	2.964	4.278	1.056	0.309	1.281
1.912	3.924	2.19	3	4.305	3.376	2.246	3.699	1.432	2.097	2.934	4.24	1.48
2.194	3.103	4.376	1.615	2.223	0.04	1.866	2.385	3.443	0.301	1.876	2.481	3.467
4.663	2.085	2.89	2.038	2.82	1.124	1.981	2.661	3.779	3.114	4.449	1.619	2.224
3.117	4.485	1.652	2.229	3.166	4.57	1.652						

Table 6.3: **Failure Times Data of air conditioning system of an aircraft**

23	261	87	7	120	14	62	47	225	71	246	21	42
20	12	120	11	3	71	11	14	11	16	90	1	16
52	95	14	5									

Table 6.4: **Failure times data of service times of 63 aircraft**

0.046	1.436	1.003	2.137	2.3	3.5	1.01	2.141	3.622	1.085	2.163	2.592	0.140
1.492	2.6	0.150	1.580	2.670	0.248	1.719	2.717	2.820	0.389	1.920	0.313	1.915
1.52	2.24	4.015	1.183	2.878	0.487	1.963	2.95	0.622	1.978	3.003	0.28	1.794
2.819	2.053	3.102	0.952	2.065	3.304	0.996	0.900	1.092	2.183	3.692	2.117	3.483
3.665	2.341	4.628	1.244	2.435	4.806	4.881	1.262	2.543	5.14			

Papers Under Review

- Anuwoje Ida L. Abonongo, Albert Luguterah, Suleman Nasiru. The Nadarajah Haghighi Generalised Power Weibull: Properties and Applications. *Journal of applied mathematics and Information Science*, Paper ID: AMIS072121A.

- Anuwoje Ida L. Abonongo1, Albert Luguterah, Suleman Nasiru. Power Series Generalised Power Weibull Class of Distribution. Journal of Statistics Applications and Probability.

Under Review