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CHEN FAMILY OF DISTRIBUTIONS WITH APPLICATIONS TO LIFETIME DATA

LEA ANZAGRA

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CHEN FAMILY OF DISTRIBUTIONS WITH APPLICATIONS TO LIFETIME DATA

BY

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(BSc. Statistics, MSc. Applied Statistics)

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DECLARATION

Student's Declaration

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree in this University or elsewhere:

Candidate's Signature:......................... Date:............................

Lea Anzagra

Supervisor's Declaration

We hereby declare that the preparation and presentation of the thesis was supervised in accordance with the guidelines on supervision of thesis laid down by the University for Development Studies.

Principal Supervisor's Signature:......................... Date:............................

Dr. Solomon Sarpong

Co-Supervisor's Signature:......................... Date:............................

Dr. Suleman Nasiru

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DEDICATION

This work is dedicated to Mum, Dad, She, Jair, Jason, Jadon and Joel.

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ABSTRACT

Classical distributions are at times unable to provide a reasonable fit to certain forms of datasets, hence the need to generalize existing distributions to enhance their flexibility in the modeling of data. In recent times, much attention is focused on developing of new families of distributions for generalizing existing models. This is evident in the vast literature on modification and generalization of statistical distributions carried out by researchers. This study therefore developed generators of statistical distributions; the odd Chen and Chen generated families of distributions, using Chen distribution as the baseline model. Statistical properties of the developed families of distributions such as the quantile functions, moments, generating functions, order statistics and entropies were derived. The parameters of the generators were estimated and special distributions developed. Properties of the estimators for the parameters of some of the special distributions were investigated using Monte Carlo simulations. The usefulness of the special distributions in modeling real dataset was then demonstrated using four datasets. The developed distributions provided good fit to the given datasets and provided consistently better fit to these datasets than the existing competing models. Finally, the new distributions developed are capable of modeling both monotonic and non-monotonic failure rates, hence it is recommended that the distributions be considered, especially in situations were datasets exhibiting such failure rates are encountered.

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ABBREVIATIONS AND ACRONYMS

AD	Anderson-Darling
AIC	Akaike Information Criterion
AICc	Corrected Akaike Information Criterion
BFGS	Broyden-Fletcher-Goldfarb-Shanno
BIC	Bayesian Information Criterion
CAIC	Consistent Akaike Information Criterion
cdf	Cumulative Distribution Function
CB	Chen Burr III
CG	Chen generated
CK	Chen Kumaraswamy
CW	Chen Weibull
CM	Cramer-von Misses
edf	empirical distribution function
EW	Generalized Weibull
EC	Exponentiated Chen
KS	Kolmogorov-Smirnov
KEC	Kumaraswamy exponentiated Chen
mle	maximum likelihood estimation
OC	odd Chen
OCB	odd Chen Burr III

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OCL odd Chen Lomax

OCW odd Chen Weibull

pdf Probability Density Function

TTT Total Time on Test

T-X Transformed-Transformer

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CHAPTER ONE

INTRODUCTION

1.1 Background of Study

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Statistical distributions are used in many disciplines, they are used in; actuarial science to model waiting time to payment of claims, reliability engineering to model the life cycle of a machine, computer science to model failure rate of system hardware or software, social science to model the average time to passing of judgment on court cases and medical science to model the survival times of patients after surgery.

The accuracy of parametric statistical inference and modeling of datasets largely depend on how well the probability distribution fits the given dataset once it has met all distributional assumptions. Weibull distribution is the most popular parametric distribution for modeling lifetime datasets (Murthy et al., 2004). However, its inability to exhibit bathtub-shaped failure rate functions is its major drawback among others, since most lifetime data tends to exhibit non-monotonic failure rates. Hence, several modifications of the Weibull distribution have been carried out overtime to make it more flexible for modeling datasets of varying shapes of hazard rate functions. One such modification is the Chen distribution (Chen, 2000) which was developed by compounding the Weibull and exponential distributions.

The Chen distribution with just two parameters has the ability to model data which exhibit increasing and bathtub shaped failure rates. Also, its confidence interval for the shape parameter and joint confidence regions for the two parameters have close forms as compared to other competing models. However, the Chen distribution has received comparatively little attention in statistical literature.

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Modifications and extensions of the Chen distribution in literature include; a study by Xie et al. (2002) which modifies the Chen distribution by adding the lacking scale parameter, thus creating a three-parameter extended Weibull (EW) distribution. Chaubey and Zhang (2015) developed an extension of the Chen distribution called the exponentiated Chen distribution using the Lehman alternatives also known as exponentiated type family (Gupta et al., 1998; Nadarajah and Kotz, 2006). The study also revealed that the exponentiated Chen distribution was a good substitute for the exponentiated Weibull and generalized Weibull families. Khan et al. (2018) proposed the Kumaraswamy exponentiated Chen (KEC) distribution. They sought to improve the statistical properties of the Chen distribution by compounding the exponentiated Chen distribution with the Kumaraswamy generalized class of distributions (Cordeiro and de Castro, 2011).

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Compounding the generalized Chen and gamma distributions, the extended Chen distribution was proposed by Bhatti et al. (2019). The study showed that the extended Chen distribution's hazard rate function could accommodate some monotonic and non monotonic shapes. The Weibull–Chen distribution was then proposed by Tarvirdizade and Ahmadpour (2019) by compounding the Weibull and Chen distributions. The new distribution was shown to be a generalization of some lifetime distributions such as exponential, Rayleigh, Weibull and Chen distributions.

In the year 2020, some studies on modifications and generalizations of the Chen distribution were published after the publication of some parts of this thesis. These include; a study by Thach and Bris (2020) on additive Chen-Weibull distribution. The distribution was developed by combining Weibull and Chen distributions for independent systems connected in series. The additive Chen-Weibull distribution was then shown to provide flexibility in modeling diverse shapes of failure rate functions.

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The study by Boateng (2020) introduced the quantile transmuted-Chen G family of distributions as a generalization of the Chen-G family of distributions (Anzagra et al., 2020a). The study indicated the usefulness of the quantile transmuted-Chen G family of distributions in modeling breast cancer patient's data. The exponentiated odd Chen-G family of distributions was proposed by Eliwa et al. (2020b). Their study established the statistical properties of the generator and then estimated its parameters using various estimation techniques.

Finally, in an independent study, Eliwa et al. (2020a) proposed and studied the odd Chen generator of distributions. It must be noted that the approach used in developing the generator and some of the special distributions in their study is the same as those used in developing the odd Chen family of distributions in this thesis. The only variation of the study from the odd Chen family of distributions (Anzagra et al., 2020b) studied in this thesis is the fact that statistical properties such as the entropies and moment generating functions which were derived in this thesis were not captured by Eliwa et al. (2020a). Furthermore, some of their developed distributions were different from those developed in this thesis.

This study sought to develop generators of the Chen distribution using the transformed-transformer (T-X) approach with the aim of improving its flexibility in the modeling of datasets.

1.2 Problem Statement

Advancement of research in various fields have resulted in real life data which sometimes cannot be modeled using any of the existing classical probability models. Hence, research in developing new classes of distributions which are generalizations or extensions of others, geared towards improving the flexibility of existing distributions remain very essential.

Despite its desirable properties, the lack of scale parameter in the Chen distribution makes it less flexible for modeling varying lifetime data, as it can only model

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datasets that exhibit bathtub-shaped or increasing failure rates (Xie et al., 2002). Though there are modifications and extensions of Chen distribution in statistical literature, until recently, these studies did not focus on developing generators based on the Chen distribution.

The study therefore sought to develop generalizations of the Chen distribution using the T-X approach. The new distributions obtained from these generalizations are expected to have at least a scale parameter and extra shape parameters to make them more flexible comparatively. They are expected to model monotonic, non-monotonic and modified non-monotonic failure rate functions.

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1.3 General Objective

The aim of the study is to develop generators of statistical distributions based on Chen distribution and apply them in modeling datasets.

1.4 Specific Objectives

The specific objectives are;

1. To develop the odd Chen family of distributions.
2. To develop the Chen generated family of distributions.
3. To study the properties of the proposed families.
4. To develop special distributions from these families of distributions.
5. To demonstrate the application of the special distributions developed from these families using real data.

1.5 Significance of the Study

The importance of statistical probability distributions in theory and practice cannot be over emphasized since their applications stretch far and wide across almost

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all disciplines (engineering, medical sciences, actuarial sciences, social sciences and so on). Statistical probability distributions are the backbone of parametric statistical methods such as reliability analysis, survival analysis and inference among others.

Fitting the appropriate distribution to lifetime data improves precision of the results of the analysis by improving the power, efficiency and sensitivity of the tests associated with the dataset. Thus, developing modifications of existing distributions to ensure better fit for datasets is of great essence. Hence, this study generalizes the Chen distribution using the T-X approach.

1.6 Scope of the Study

This study mainly focuses on developing generators with Chen distribution as the baseline model using the T-X approach. It also considers deriving the properties of the proposed generator and demonstrating the usefulness of these generators using real life datasets.

1.7 Outline of Thesis

The thesis is divided into six chapters. Introduction is in Chapter 1. Literature on the methods of developing new distributions is reviewed in Chapter 2. Some statistical techniques and tools used for achieving the aim of the study are discussed in the methodology in Chapter 3. The odd Chen family of distributions is presented in Chapter 4, whilst the Chen generated family of distributions is presented in Chapter 5. Finally, the summary, conclusions and recommendations of the study are presented in chapter 6.

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CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Literature on the proposed methods for developing new distributions is reviewed in this chapter. The development of statistical distributions has always been topical, hence research on methods of developing new distributions dates' way back. Advances in methods of developing new distributions can at best be categorized into two; methods before the 1980s and methods since the 1980s. Before the 1980s, the proposed methods may be broadly categorized into three; differential equation, transformation and quantile function. Those developed after the 1980s may also be generally classified into four categories; the T-X method, beta-generated method, method of adding parameters to existing distributions and method of generating skewed distributions (Lee et al., 2013).

2.2 Method of Differential Equations

Pearson (1895) made huge contributions towards the development of this method. In an effort to model non-symmetric data, he proposed the use of differential equations for generating statistical distributions. Per the approach, every probability density function (pdf) in a system of continuous distributions satisfy the differential equation. Based on the shape of the pdfs of these distributions, they were then classified into types, thus Pearson types I-IV, and Pearson types VII-XII in a later study.

Many of the classical statistical distributions are derived from the Pearson type distributions. These include: beta distribution (Pearson type I), normal and Student's T distributions (Pearson type VII) and gamma distribution (Pearson

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type III) (Johnson et al., 1994). Burr (1942) also made a significant contribution by proposing another form of differential equations for developing probability distributions. Distributions derived from this family include; Burr III, Burr X, Burr XII and uniform distributions.

2.3 Method of Transformation (or Translation)

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Also referred to as the method of translation, Johnson (1949) proposed this method based on the use of normalization transformation. Commonly used distributions such as gamma, exponential, normal, lognormal and beta distributions are members of the Johnson's family. A special distribution from the Johnson's family for modeling material fatigue is the Birnbaum-Saunders distribution (Birnbaum and Saunders, 1969).

Some modifications of Birnbaum-Saunders distribution; families of location-scale Birnbaum-Saunders, non-central Birnbaum-Saunders and four parameter generalized Birnbaum-Saunders distributions among others, were then made using Johnson's approach (Athayde et al., 2012).

2.4 Method of Quantile Function

Lambda distribution was developed using the quantile approach (Hastings et al., 1947; Tukey, 1960). It was then generalized as the generalized lambda distribution and defined in terms of percentile functions (Ramberg and Schmeiser, 1972; Ramberg et al., 1979).

Though it shares similarities with the Pearson's system, the generalized lambda distribution had a weakness of not covering all skewness and kurtosis values (Freimer et al., 1988). Hence, the extended generalized lambda distribution developed from generalized beta and generalized lambda distributions by Karian and Dudewicz (2000) was to overcome that weakness. Some works carried out using the idea of quantiles are found in Tuner and Pruitt (1978), Morgenthaler

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and Tukey (2000) and Jones (2002).

2.5 Method of Generating Skewed Distributions

Skewed distributions were formed by combining two symmetric distributions. This approach of generating skewed distributions is attributed to Azzalini (1985) who proposed the skewed normal family. The initial idea of this approach was used in the context of prior distribution by O'Hagan and Leonard (1976). The proposed skewed normal family only produced thinner tails compared with the normal ones hence a much broader class of distributions was later proposed by Azzalini (1986).

Ever since, extensive studies on the skewed family have been carried out and many generalizations developed. Using the framework of Azzalini (1986), Chang and Genton (2007) proposed a weighted approach to generating distributions from skewed symmetric family. The epsilon-skew normal family was then developed by Mudholkar and Hutson (2000). This family has additional parameter which controls the magnitude of skewness. Salinas et al. (2007) developed a broad family of skewed distributions by combining the epsilon-skew normal and the skew normal families together.

Fernandez and Steel (1998) introduced the inverse scale family. This method introduced skewness into unimodal and symmetric continuous distributions. The single scalar parameter in their approach creates flexibility in the distribution's shape whilst maintaining the distribution's unimodality. Using inverse probability integral transformation, a generalized framework of adding skewness into symmetric distributions was proposed by Ferreira and Steel (2006). Members of this family include the skewed normal and the inverse scale families of distributions.

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2.6 Method of Adding Parameters

This approach involves the addition of parameters to existing distributions to increase their flexibility in the modeling of data. Though this method had been in use, the study by Mudholkar and Srivastava (1993) on exponentiated Weibull distribution brought it to the lime light (Lee et al., 2013). Other distributions were then proposed and studied using the approach; Gupta and Kundu (1999, 2001) proposed the exponentiated exponential distribution and Nadarajah and Kotz (2006) studied a number of exponentiated distributions such as exponentiated exponential, exponentiated gamma, exponentiated Weibull, exponentiated Gumbel and exponentiated Fréchet distributions. Another method of adding an extra parameter to distributions was introduced by Marshall and Olkin (2007). The approach was then applied in studying the case of exponential and Weibull distributions.

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2.7 Beta-Generated Method

First proposed by Eugene et al. (2002), the beta-generated family of distributions can be described as a generalization of distributions using beta distribution as its generator (Jones, 2009). Some beta generated distributions proposed in literature include: beta-normal (Eugene et al., 2002); beta-Frechet (Nadarajah and Gupta, 2004); beta-Gumbel (Nadarajah and Kotz, 2004); beta-exponential (Nadarajah and Kotz, 2005); beta-Weibull (Famoye et al., 2005); beta-exponentiated Pareto (Zea et al., 2012); beta- Cauchy (Alshawarbeh et al., 2012); beta-extended Weibull (Cordeiro et al., 2012) and beta- generalized logistic (Morais et al., 2013).

In literature, generalized versions of the beta-generated families have been developed by changing the beta distribution with distributions defined on a finite support. The Kumaraswamy generated family of distributions independently proposed by Jones (2009) and Cordeiro and de Castro (2011), was obtained by using Kumaraswamy distribution (Kumaraswamy, 1980) as the generator in place of

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beta distribution. Another generalization of the beta-generated family was introduced by Alexander et al. (2012). They used generalized beta type-I distribution as the generator.

2.8 Transformed-Transformer Method

Using the idea of the beta-generated method, Alzaatreh et al. (2013) proposed the T-X family of distributions. This approach generalizes the beta-generated method and uses any continuous distribution as its generator. Generators developed from this family include; Weibull-X, Gamma-X and beta- exponential-X families. The major drawback of the T-X method is its lack of an in built shape parameter. Hence if two distributions to be compounded both lack shape parameters (say both T and X follow exponential distribution), then the resulting distribution would still lack a shape parameter. Hence it would have failed to achieve its aim of producing a flexible distribution.

Alzaghal et al. (2013) introducing a shape parameter to the T-X family, proposed the exponentiated T-X family. Some members of the exponentiated T-X family include: exponentiated gamma-X, exponentiated Weibull-X, exponentiated Lomax-X and exponentiated log-logistic-X families. The major limitation of the exponentiated T-X family is its inability to produce distributions with heavy tails, especially, when the baseline distribution lack shape parameters. Generally, additional shape parameters improve the flexibility of a model as it controls both skewness and kurtosis simultaneously.

To further improve on the flexibility of the exponentiated T-X family, Nasiru et al. (2017) introduced an extra shape parameter in a generalization of the exponentiated T-X family called generalized exponentiated T-X family. Though this family may produce much more flexible distributions, it also has the tendency of producing over-parameterized distributions especially for baseline distributions with two or more parameters.

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2.9 Summary of Review

It is almost impossible to develop a distribution that is flexible enough to fit all forms of data, hence researchers keep developing new distributions using these methods irrespective of the time of origin in their bid to develop distributions with desirable properties. Among the many methods for generalizing distributions, the T-X method provides greater flexibility in the generalization and modification of distributions especially when the baseline model already has shape parameters.

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CHAPTER THREE

METHODOLOGY

3.1 Introduction

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The statistical techniques and tools used for achieving the aim of the study are discussed in this chapter. These include; the *T*-*X* method for developing the generator, maximum likelihood estimation method for estimating the parameters of the new family, the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm for optimization, goodness-of-fit and information criteria measures for model fit, and total time on test transform.

3.2 The T-X Method

Suppose $z(t)$ is the pdf of a random variable $T \in [a, b]$ for $-\infty \leq a < b \leq \infty$. Let $G(x)$ be the cdf of any random variable X, the cdf of the T-X family is given by

$$F(x) = \int_{a}^{W[G(x)]} z(t)dt, \tag{3.1}$$

where

1. $W[G(x)]$ is a function of the cdf of any random variable X which is differentiable, monotonically non-decreasing and defined on the support $[a, b]$.

2. $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

3.3 Maximum Likelihood Estimation

The maximum likelihood estimation (mle) approach seeks the probability distribution that makes the observed data most likely. Let $X_1, X_2, ..., X_n$ be n independent and identically distributed random variables with a common pdf $f(x; \varphi)$

where φ is an unknown $(p \times 1)$ vector of parameters. The likelihood function which is the joint pdf of a collected random sample and the basis of the mle procedure is obtained as

$$L(\varphi; X) = \prod_{i=1}^{n} f(x_i; \varphi), \tag{3.2}$$

where $X = (x_1, x_2, ..., x_n)$ and $p < n$ (Hogg et al., 2005). Mathematically, working with the logarithm of the likelihood function is more convenient and does not lead to any loss of information. Hence, denoting the log likelihood function by ℓ, it is obtained as

$$\ell(\varphi; X) = \sum_{i=1}^{n} \log f(x_i; \varphi). \tag{3.3}$$

The values of φ that maximize the probability of obtaining the random sample is obtained by differentiating ℓ with respect to φ and equating the resultant expression to zero. Thus

$$\frac{\partial \ell(\varphi; X)}{\partial \varphi_i} = 0, i = 1, 2, ..., p. \tag{3.4}$$

The maximum likelihood estimates $\hat{\varphi}$ for the parameters are the values of φ that maximize the likelihood function and are obtained by solving the equations in (3.4) for $\varphi_1, \varphi_2, ..., \varphi_p$. The Fishers information matrix I, which is used in generating the variance-covariance matrix of the estimators, is generated using the mle approach. The variance covariance matrix is the inverse of I obtained as

$$I^{-1}(\varphi) = \begin{bmatrix} I_{11}^{-1} & I_{12}^{-1} & I_{13}^{-1} & \cdots & I_{1p}^{-1} \\ I_{21}^{-1} & I_{22}^{-1} & I_{23}^{-1} & \cdots & I_{2p}^{-1} \\ I_{31}^{-1} & I_{32}^{-1} & I_{33}^{-1} & \cdots & I_{3p}^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{p1}^{-1} & I_{p2}^{-1} & I_{p3}^{-1} & \cdots & I_{pp}^{-1} \end{bmatrix}, \tag{3.5}$$

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where the I_{ij} element of I is given by

$$I_{ij} = -E\left[\frac{\partial^2 \ell(\varphi; X)}{\partial\varphi_i \partial\varphi_j}\right],$$

$I_{ii}^{-1} = var(\hat{\varphi})$ is the variance of $\hat{\varphi}_i$ and $I_{ij}^{-1} = cov(\hat{\varphi}_i, \hat{\varphi}_j)$ is the covariance of $\hat{\varphi}_i$ and $\hat{\varphi}_j$. Using the estimated variances of the parameters, the approximate $100(1-\alpha)\%$ confidence interval for the parameters in normal approximation is estimated as $\varphi_i \in \hat{\varphi}_i \pm Z_{\frac{\alpha}{2}}\sqrt{I_{ii}^{-1}}$, where $Z_{\frac{\alpha}{2}}$ is the critical value from the standard normal distribution.

3.3.1 Properties of Maximum Likelihood Estimation

Under certain regularity conditions (such as the assumption that the pdfs have a common support for all φ_i, the random variables have distinct pdfs $f(x;\varphi_i)$ such that $\varphi_i \neq \varphi_j \Rightarrow f(x;\varphi_i) \neq f(x;\varphi_j)$ and the true value of the population parameter φ is an interior point in φ), the maximum likelihood estimators have desirable properties such as consistency, asymptotic normality, asymptotic efficiency and invariance property.

3.3.1.1 Consistency

Let $X_1, X_2, ..., X_n$ be a sequence of observations with an estimator $\hat{\varphi}_n$. $\hat{\varphi}_n$ is a consistent estimator if it converges asymptotically in probability to the true value of the population parameter. Thus as $n \longrightarrow \infty$,

$$P(|\hat{\varphi}_n - \varphi| \geq \varepsilon) \to 0, \varepsilon > 0. \tag{3.6}$$

Also, $\hat{\varphi}_n$ converges in probability to φ if the mean squared error goes to zero as n approaches infinity, thus, $\lim_{n\to\infty} E[(\hat{\varphi}_n - \varphi)^2] = 0$. Hence, as sample size increases, the maximum likelihood estimators converge to the true parameter value (Hogg et al., 2005).

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3.3.1.2 Asymptotic Normality

The distribution of maximum likelihood estimators under certain regularity conditions converges to multivariate normal distribution as sample size increases. Thus

$$\sqrt{n}(\hat{\varphi} - \varphi) \rightarrow N(0, I^{-1}(\varphi)), \tag{3.7}$$

where $\rightarrow$ represents convergence in distribution, 0 is the p-dimensional mean zero vector and $I^{-1}(\varphi)$ is the inverse of the $(p \times p)$ dimensional Fisher information matrix.

3.3.1.3 Invariance Property

Maximum likelihood estimation is functional under all transformations. Thus for a differentiable function $f(\varphi)$, the maximum likelihood estimate of $f(\varphi)$ is equal to the function evaluated at the maximum likelihood estimation of φ, implying that, if $\hat{\varphi}$ is the maximum likelihood estimate of φ, then $f(\hat{\varphi})$ is the maximum likelihood estimate of $f(\varphi)$. Hence,

$$\sqrt{n}(f(\hat{\varphi}) - f(\varphi)) \rightarrow N\left(0, \left[\frac{\partial f(\varphi)}{\partial \varphi}\right] I^{-1}(\varphi) \left[\frac{\partial f(\varphi)}{\partial \varphi}\right]^{\prime}\right). \tag{3.8}$$

3.3.1.4 Asymptotic Efficiency

Maximum likelihood estimators are asymptotically most efficient. A consistent estimator is most efficient if it has the minimum variance in a class of unbiased and consistent estimators (Hogg et al., 2005). For instance, given an alternative unbiased estimator $\bar{\varphi}$, such that

$$\sqrt{n}(\bar{\varphi} - \varphi) \rightarrow N(0, I^{-1}(\Psi)). \tag{3.9}$$

then $I^{-1}\varphi$ is always less than or equal to $I^{-1}(\Psi)$.

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3.3.1.5 Broyden–Fletcher–Goldfarb–Shanno Algorithm

The Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm is one of the quasi-Newton iterative approach for resolving unconstrained optimization problems named after Broyden (1970), Fletcher (1970), Goldfarb (1970) and Shanno (1970) who independently developed it. It is useful when solving equations generated from the mle process especially when the estimators for the parameters have no closed form. The algorithm for optimizing a function say ℓ starts with an initial guess φ_0 and an approximate Hessian matrix H_0. As φ_i converges to the solution, these steps are usually repeated:

1. Solve $H_i c_i + \nabla\ell(\varphi_i) = 0$ to obtain a direction .
2. Carry out one-dimensional optimization to obtain an acceptable step size d_i in the same direction found in step one.
3. Let $b_i = c_i d_i$ and update $\varphi_{i+1} = \varphi_i + b_i$.
4. Set $y_i = \nabla\ell(\varphi_{i+1}) + \nabla\ell(\varphi_i)$.
5. $H_{i+1} = H_0 + \frac{y_i y_i'}{y_i' b_i} + \frac{H_i b_i b_i' H_i}{b_i' H_i b_i}$.

Let $\ell(\varphi)$ be a function to be minimized, by observing the norm of the gradient $|\ell(\varphi_i)|$, the convergence of the algorithm can be checked. Step one approximates to a gradient descent when H_0 is initialized with the identity matrix I, however approximation of the Hessian H_i results in the refinement of further steps. Step one of the algorithm is performed out using the inverse of H_i. Step one can also be efficiently obtained by transforming the fifth step using Sherman–Morrison formula

$$H_{i+1}^{-1} = \left(I - \frac{b_i' y_i}{b_i y_i'}\right) H_i^{-1} \left(I - \frac{b_i' y_i}{b_i y_i'}\right) + \frac{b_i b_i'}{b_i y_i'}. \tag{3.10}$$

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Since $y_i' H_i^{-1} y_i$ and $b_i' y_i$ are scalars and H_{i+1}^{-1} is symmetric, H_{i+1}^{-1} can be computed using the expansion

$$H_{i+1}^{-1} = H_i^{-1} + \frac{\left(b_i' y_i + y_i' H_i^{-1} y_i\right)\left(b_i b_i'\right)}{\left(b_i' y_i\right)^2} - \frac{H_i^{-1} y_i b_i' + b_i y_i' H_i^{-1}}{b_i' y_i}. \tag{3.11}$$

Confidence intervals for parameters in maximum likelihood estimation and other statistical estimations can be obtained using the inverse of the final Hessian matrix.

3.4 Goodness-of-Fit Tests

A goodness-of-fit test generally examines how well a dataset corresponds to a fitted distribution. For a random sample $X_1, X_2, ..., X_n$, the goodness-of-fit test determines if the random sample is from a specific distribution. Anderson-Darling (AD) test, Kolmogorov-Smirnov (KS) test and Cramer-von Misses (CM) test are used in the study.

3.4.1 Kolmogorov-Smirnov Test

Let $X_1, X_2, ..., X_n$ be a random sample. The KS test is based on the empirical distribution function (edf) as it measures the distance between the estimated cdf and the edf of the sample (Chambers et al., 1983). The hypothesis that the data follow a specified distribution is tested against the alternate that it does not. Its test statistic is given by

$$KS = max\{|F(x_i) - \hat{F}(x_i)|, |F(x_i) - \hat{F}(x_{i-1})|\}, i = 1, 2, ..., n, \tag{3.12}$$

where $F(x_i) = \frac{r\{x_j : x_j \leq x_i\}}{n}$ is the cdf of the candidate model at x_i, $\hat{F}(x_i)$ is the value of the empirical distribution at x_i and $r\{.\}$ is the total number observations with values less than or equal to that of x_i. The decision on the rejection (or non-rejection) of the null hypothesis is then made based on a comparison between

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the computed test statistic and the critical value obtained for a given significance level (α). In comparing two or more models, the one with the smallest KS value provides the most appropriate fit for the sample.

3.4.2 Anderson-Darling Test

The Anderson-Darling (AD) test is an alternative and a modification of the KS test. The main difference is that whilst KS test is distribution free, AD test is not. This makes the AD test comparatively more sensitive (Stephens, 1974). Its test statistic is defined as

$$AD = -N - S, \tag{3.13}$$

where $S = \sum_{i=1}^{N}(2i-1)N\left[\ln F(X_i) + \ln\left(1 - F(x_{N+1-i})\right)\right]$, N is the sample size, X_i's are the ordered observations and $F(X_i)$ is the cdf of the candidate model at X_i. Generally, when comparing two or more models, the model with the smallest AD value should be considered.

3.4.3 Cramér-von Mises Test

The Cramér-von Mises (CM) test is another alternative to the KS test (Laio, 2004). Let $F(X_i;\varphi)$ be the cdf of the random sample with an unknown $p-$dimensional parameter vector. The quantile can be estimated by finding the inverse of $F(X_i;\varphi) = u_i$. With the empirical distribution forming its basis, the test statistic for the CM test is defined by

$$CM = W^2\left(1 + \frac{0.5}{n}\right), \tag{3.14}$$

where $W^2 = \sum_{i=1}^{n}\left(z_i - \frac{(2i-1)}{2n}\right)^2 + \frac{1}{12n}$, n is the sample size, X_i's are the ordered observations, $z_i = \Phi^{-1}(u_i)$ is the standard normal distribution's quantile and $\Phi(u_i)$ is the cdf. The model with the smallest CM value fits the sample better as compared to the others.

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3.5 Information Criteria

Generally, in model selection for data analysis, one is faced with the challenge of balancing the issues of variance and bias, thus overfitting and underfitting. A model with more parameters is generally less biased with high variance, whilst one with fewer parameters may be highly biased with a low variance. Hence, the goal is to select a fitted model with minimized information loss. Coverage of parameter's confidence interval in a model shows if models are properly selected or not. Information criterion compares non-nested models by ordering candidate models from best to worst. It then scales these models using Akaike weights and evidence ratios. Some common information criteria are; the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC).

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3.5.1 Akaike Information Criterion

AIC can be described as an estimated measure of the quality of a model in a set of competing statistical models for a particular dataset. Developed with a foundation in information theory, AIC was first proposed by Akaike (1973); its test statistic is given by

$$AIC = -2\ell + 2p, \tag{3.15}$$

where p is the number of estimated model parameters and ℓ is the log-likelihood of the model fitted. ℓ is also a measure of model fit, the higher its value, the better the fit. AIC gives an estimate of the amount of information lost due to fitting a particular model to a dataset. Hence, a model that yields the smallest AIC value in the set of competing models is deemed the best model. AIC introduces good model selection especially for large samples as it is able to penalize models with many parameters. It is however associated with issues of bias especially for smaller samples, hence the 'corrected' Akaike Information Criterion (AICc) was developed to overcome the problem (Sugiura, 1978). The test statistic for the

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AICc is

$$AIC_c = -2\ell + 2p + [2p(p+1)/(n-p-1)]\,, \tag{3.16}$$

where n is the sample size.

3.5.2 Bayesian Information criterion

The BIC also known as Schwarz criterion is closely related to the AIC as it has likelihood function as its basis for computation. Like the AIC, BIC penalizes over parameterization (over fitting) of models. Compared with AIC and AICc, models with more parameters are more penalized by BIC, hence the need to consider BIC in model selection. Introduced by Schwarz (1978), the best model of the set of competing model is the one with the smallest value of BIC. The test statistics for the BIC is given by

$$BIC = -2\ell + p\log n, \tag{3.17}$$

where ℓ is the log-likelihood of the model fitted, p is the number of estimated model parameters and n is the sample size.

3.5.3 Consistent Akaike Information Criterion

CAIC is one of the 'dimension-consistent' criteria for model selection reviewed by Bozdogan (1987). It provides an asymptotically unbiased estimate of the order of the true model on the assumption that there exists a true model of low order, whose order remains same as sample size increases (Anderson et al., 1998). The test statistic of the CAIC is given by

$$CAIC = -2\ell + p[(\log n) + 1]. \tag{3.18}$$

Prediction is mostly the modeling goal in the use of CAIC selection. In case the true model does not exist, CAIC should not be applied.

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3.6 Total Time on Test

The total time on test transform (TTT- transform) gives a graphical presentation of the shape of hazard rate function of a given dataset developed by Barlow and Doksum (1972). Let F be the cdf of a distribution, the TTT- transform is defined as

$$G^{-1}(k) = \int_0^{F^{-1}(k)} S(u)du, k \in [0,1], \tag{3.19}$$

where $S(u) = 1 - F(u)$ is the survival function. The scaled TTT-transform is obtained by $\vartheta_F(k) = \frac{G^{-1}(k)}{G^{-1}(1)}$.

Graphically, the hazard rate function is obtained by a plotted curve of $\vartheta_F(k)$ against k called the scaled TTT-transform curve. The resultant curve may then assume shapes such as decreasing, increasing, bathtub or upside down bathtub. The empirical scaled TTT-transform for an ordered random sample $X_{1:n}, X_{2:n}, ..., X_{n:n}$ of size n is obtained by

$$T_i^* = \frac{TTT_i}{TTT_n}, \tag{3.20}$$

where $0 \leq TTT_n \leq 1$ and $TTT_i = \sum_{j=1}^{i}(n-j+1)(x_{j:n} - x_{j-1:n}), i = 1, 2, ..., n.$ By plotting T_i^* against $\frac{i}{n}$, the empirical scaled TTT-transform curve is obtained.

3.7 Data and Source

The study employed four secondary datasets (displayed in Appendix A of the thesis) in demonstrating the applications of the distributions developed. The first dataset (Data 1) consists of the lifetimes of 50 components, given by Aarset (1987). This data have been used in many studies especially studies on statistical distributions. Some studies that have used this dataset include; Tarvirdizade and Ahmadpour (2019) and Doostmoradi et al. (2014). The second dataset (Data 2) represents the lifetime of a certain device given by Sylwia (2007) and also found in Doostmoradi et al. (2014). The third dataset (Data 3) consists the fatigue

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times of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second found in Birnbaum and Saunders (1969). The fourth dataset (Data 4) consists the survival times (in days) of 72 guinea pigs injected with different amount of virulent tubercle bacilli studied by Bjerkedal (1960).

3.8 Summary

The statistical techniques to be employed in the study are thoroughly reviewed under this chapter. These include maximum likelihood estimation and its properties, measures of goodness-of-fit and information criteria for model selection. BFGS algorithm and total time on test were also reviewed.

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CHAPTER FOUR

ODD CHEN FAMILY OF DISTRIBUTIONS

4.1 Introduction

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This chapter introduces a new generalization of the Chen distribution called the odd Chen (OC) family of distributions. This generator can be used to modify any univariate continuous distribution to improve upon its flexibility in modeling datasets. Statistical properties such as the quantile function, moments, stochastic orderings and order statistics among others for the OC family of distributions are also derived in this chapter. The chapter further presents the estimation of parameters of the new family, the development of new distributions from the new family and finally, a demonstration of the usefulness of the new models in modeling real dataset.

4.2 Odd Chen Family of Distributions

Let T be a Chen distributed continuous random variable, the cdf (denoted by $F(t)$) for Chen distribution is given by $F(t) = 1 - e^{\lambda\left(1-e^{t^{\beta}}\right)}, t > 0$ (Chen, 2000). Suppose $G(x;\psi)$ is the baseline cdf of an arbitrary continuous random variable X on any continuous support say $(-\infty, \infty)$ and ψ is a $(p \times 1)$ vector of associated parameters, the cdf $F(x)$ of the OC family of distributions is defined as

$$F(x) = \int_{0}^{\frac{G(x;\psi)}{1-G(x;\psi)}} f(t)dt = 1 - e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}, x > 0, \lambda > 0, \beta > 0, \qquad (4.1)$$

where λ and β are extra shape parameters.

By differentiating the cdf in equation (4.1), the pdf $f(x)$ of the family is obtained

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as

$$f(x) = \lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1-G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}} \times e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}, x > 0. \tag{4.2}$$

Proposition 4.1. The density function of the OC family of distributions is a well-defined pdf.

Proof. The pdf $f(x)$ of a distribution is well-defined if it is a non-negative function and is equal to one(1) when integrated over the support of X. Thus $f(x)$ is well-defined if

$$\begin{cases} f(x) \geq 0 \\ \int_{-\infty}^{\infty} f(x)dx = 1, -\infty \leq x \leq \infty \end{cases}.$$

It is worth noting that $f(x)$ is a non-negative function. Suppose the support of x is $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1-G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}} \times e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)} dx.$$

Let

$$u = e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)},$$

then

$$u = \begin{cases} 1, & x \to -\infty \\ 0, & x \to \infty \end{cases}.$$

Also,

$$\frac{du}{dx} = \lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1-G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)},$$

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implying that

$$dx = \frac{du}{\lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1-G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}}.$$

Hence,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{1} du = 1.$$

This completes the proof.

The survival and failure rate functions play a pivotal role in reliability analysis and other disciplines. Survival (or reliability) function gives the probability of performing without fail a specified task, under given conditions for a specific period of time. Thus, reliability may be used as a measure of the system's success in proforming its function properly. Mathematically, the survival function, $S(x)$, is expressed as

$$S(x) = 1 - F(x)$$

The failure rate (hazard) function, $h(x)$, on the other hand is the instantaneous failure rate and is mathematically expressed as

$$h(x) = \lim_{\triangle x \to 0} \frac{P(x < X \leq x + \triangle x \mid X > x)}{\triangle x} = \frac{f(x)}{1 - F(x)}$$

The survival $S(x)$ and hazard $h(x)$ functions of the OC family are respectively given by

$$S(x) = e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}, x > 0 \tag{4.3}$$

and

$$h(x) = \lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1-G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}, x > 0 \lambda > 0, \beta > 0. \tag{4.4}$$

For simplicity, let $G(x;\psi)$ and $g(x;\psi)$ be denoted as $G(x)$ and $g(x)$.

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4.3 Mixture Representation

The mixture representation of the pdf is essential in the derivation of the statistical properties of the OC family of distributions.

Proposition 4.2. The mixture representation of the pdf of the OC family is obtained as

$$f(x) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} g(x) G(x)^q, \tag{4.5}$$

where

$$\nu_{ijkmq} = \frac{(-1)^{i+m}\lambda^i (i+1)^j}{i!j!} e^{\lambda} \begin{pmatrix} \beta(j+1)+k \\ k \end{pmatrix} \begin{pmatrix} \beta(j+1)+k-1 \\ m \end{pmatrix} \begin{pmatrix} m \\ q \end{pmatrix}.$$

Proof. After applying Taylor series expansion,

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

to the pdf of the OC family in equation (4.2), $f(x)$ becomes

$$f(x) = \lambda\beta g(x) e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i\lambda^i}{i!} \frac{(i+1)^j}{j!} G(x)^{\beta(j+1)-1}[1-G(x)]^{-[\beta(j+1)+1]}.$$

Further expanding the expression for $f(x)$ using the generalized binomial series expansion,

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \begin{pmatrix} a+k-1 \\ k \end{pmatrix} z^k, |z| \leq 1, a < 0,$$

yields

$$f(x) = \lambda\beta g(x) e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i\lambda^i}{i!} \frac{(i+1)^j}{j!} \begin{pmatrix} \beta(j+1)+k \\ k \end{pmatrix} G(x)^{\beta(j+1)+k-1}.$$

The mixture representation of the pdf of the OC family $f(x)$ can further be

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expressed as

$$f(x) = \lambda\beta e^{\lambda} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \frac{(-1)^{i+m}\lambda^{i}}{i!}\frac{(i+1)^{j}}{j!}\binom{\beta(j+1)+k}{k} \times \binom{\beta(j+1)+k-1}{m}\binom{m}{q} g(x)G(x)^{q},$$

hence the proof.

Equation (4.5) expresses the pdf of the OC family as a product of its parameters and sum of the product of the pdf and weighted power series of the baseline distribution function.

Also, expressing $f(x)$ in terms of exponentiated-G (expo-G) density yields

$$f(x) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu^{*}_{ijkmq}\pi_{q+1}(x), \tag{4.6}$$

where $\nu^{*}_{ijkmq} = \frac{\nu_{ijkmq}}{q+1}$ and $\pi_{q+1}(x) = (q+1)g(x)G(x)^{q}$ is the expo-G density function with power parameter $(q+1)$.

4.4 Statistical Properties

This section discusses some of the statistical properties of the CG family of distributions. These include: quantile functions, non-central moments, moments, generating functions, inequality measures, entropies, residual life, stochastic ordering and order statistics.

4.4.1 Quantile Function

Random number generation for simulation purposes is one of the essential uses of the quantile function. Also, the effect of parameters on the skewness and kurtosis of a distribution can be determined based on the quantile measure.

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Proposition 4.3. The quantile function for the OC family of distributions is given by

$$Q_G(u) = G^{-1}\left[\frac{\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}{1+\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}\right], 0 \leq u \leq 1. \tag{4.7}$$

Proof. The quantile function $Q_G(u)$ of a random variable X, $0 \leq u \leq 1$, is defined as the inverse of the cdf $F(x)$. Replacing x with x_u in equation (4.1), equating $F(x_u)$ to u and making x_u the subject yields the quantile function. The median of the family is obtained by substituting $u = 0.5$ in equation (4.7).

The measures of skewness and kurtosis can be computed based on the quantile measures. The Bowley's measure of skewness and the Moors' measure of kurtosis are respectively defined as

$$\text{skewness} = \frac{Q(1/4) + Q(3/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)} \tag{4.8}$$

and

$$\text{kurtosis} = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(3/4) - Q(1/4)}. \tag{4.9}$$

4.4.2 Moments, Moment Generating Functions and Incomplete Moments

In this section, the moments, moment generating function and incomplete moments are derived.

4.4.2.1 Moments

Moments are useful in statistical analysis especially in the study of the characteristics of distributions such as measures of central tendencies, skewness and kurtosis.

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Proposition 4.4. The r^{th} non-central moment for the OC family of distributions is given by

$$\mu_r^{'} = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\tau_{(r,q)}, r = 1, 2, ..., \tag{4.10}$$

where $\tau_{(r,q)} = \int_{-\infty}^{\infty} x^r g(x)(G(x))^q dx$ is the weighted moment of the baseline distribution $G(x)$.

Proof. The r^{th} non-central moment is defined as

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx, r = 1, 2, ...$$

Substituting the mixture form of the density in equation (4.5) into the definition of $\mu_r^{'}$ yields

$$\mu_r^{'} = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{\infty} x^r g(x)(G(x))^q dx,$$

thus completes the proof.

Alternatively, let $G(x) = u$, $x = G^{-1}(u) = Q_G(u)$, $\frac{du}{dx} = g(x)$ and $g(x)dx = du$. The r^{th} non-central moment is defined in terms of the quantile function as

$$\mu_r^{'} = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{0}^{1} u^q Q_G^r(u)du, 0 < u < 1. \tag{4.11}$$

4.4.2.2 Moment Generating Function

Moment generating functions are functions that can be used to establish the moments of a random variable about a point. The moment generating function of the OC family of distributions if it exist is given by Proposition 4.5.

Proposition 4.5. The moment generating function for the OC family of distributions is given by

$$\mathrm{M}_X(t) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{r=0}^{\infty} \frac{t^r}{r!}\nu_{ijkmq}\tau_{(r,q)}. \tag{4.12}$$

Proof. Generally, the moment generating function for a random variable X is defined as

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx.$$

Hence, expanding $M_X(t)$ using Taylor series expansion yields

$$\mathrm{M}_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty} x^r f(x)dx.$$

Subsequently, substituting the expression for the r^{th} non-central moment, $\mu_r^{'}$, in equation (4.10) into the definitin of $M_X(t)$ yields

$$\mathrm{M}_X(t) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{r=0}^{\infty} \frac{t^r}{r!}\nu_{ijkmq}\tau_{(r,q)},$$

thus the proof.

Let $G(x) = u$, $M_X(t)$ can be expressed in terms of quantile function as

$$\mathrm{M}_X(t) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{r=0}^{\infty} \nu_{ijkmq}\int_{0}^{1} e^{tQ_G(x)}u^q du, 0 < u < 1. \tag{4.13}$$

4.4.2.3 Incomplete Moments

Incomplete moments play a key role in the computation of statistical measures such as the mean deviations about the mean and median. They are key in computing measures such as Lorenz and Bonferroni curves.

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Proposition 4.6. The incomplete moments of the OC family of distributions is given by

$$M_r(y) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{y} x^r g(x) G^q(x) dx, r = 1, 2, ... \quad (4.14)$$

Proof. For a random variable X, it's incomplete moments is defined as

$$M_r(y) = \int_{-\infty}^{y} x^r f(x) dx, r = 1, 2, ...$$

Substituting the mixture representation of the density in equation (4.5) into the expression yields

$$M_r(y) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{y} x^r g(x) G^q(x) dx,$$

hence the proof.

Let $G(x) = u$, $x = G^{-1}(u) = Q_G(u)$ and $g(x)dx = du$, the incomplete moments can be expressed in terms of quantile function as

$$M_r(y) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{0}^{G(y)} u^q Q_G^r(u) du, 0 \leq u \leq 1. \quad (4.15)$$

4.4.3 Order Statistics

Order statistics are very useful in many areas of statistical theory most especially extreme-value theory. The pdf for the p^{th} order statistic, $X_{p:n}$, of an ordered random sample, $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, of size n is denoted by $f_{X_{p:n}}(x)$.

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Proposition 4.7. The pdf for the p^{th} order statistic of the OC family of distributions is given by

$$f_{X_{p:n}}(x) = \lambda\beta \sum_{i=0}^{p-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=o}^{m} D_{ijklmn} g(x) G(x)^n,, p = 1, 2, ..., n, \quad (4.16)$$

where

$$D_{ijklmn} = \frac{(-1)^{i+j+m} n! (\lambda(n-p+i+1))^j (j+1)^k e^{\lambda(n-p+i+1)}}{j!k!(p-1)!(n-p)!} \times \binom{p-1}{i} \binom{\beta(k+1)+l}{l} \binom{\beta(k+1)+l-1}{m} \binom{m}{n}.$$

Proof. The pdf for the p^{th} order statistic, $X_{p:n}$, of a random sample, $X_1, X_2, ..., X_n$, of size n, $f_{X_{p:n}}(x)$, is generally defined as

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!} [F(x)]^{p-1} [1 - F(x)]^{n-p} f(x), p = 1, 2, ..., n.$$

Expanding $[F(x)]^{p-1}$ in the definition of $f_{X_{p:n}}(x)$ using binomial series expansion yields

$$[F(x)]^{p-1} = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} [1 - F(x)]^i.$$

Substituting it back into the expression of $f_{X_{p:n}}(x)$ yields

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} [S(x)]^{n-p+i} f(x), \quad (4.17)$$

where

$$[S(x)]^{n-p+i} = [1 - F(x)]^{n-p+i} = e^{\lambda(n-p+i)\left(1-e^{G(x)^\beta}\right)}.$$

Algebraically manipulating

$$[S(x)]^{n-p+i} f(x) = \lambda\beta g(x) G(x)^{\beta-1} e^{G(x)^\beta} e^{\lambda(n-p+i+1)\left(1-e^{G(x)^\beta}\right)}$$

using Taylor series expansion yields

$$[S(x)]^{n-p+i}f(x) = \lambda\beta g(x)G(x)^{\beta-1}e^{\lambda(n-p+i+1)} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j[\lambda(n-p+i+1)]^j(j+1)^k}{i!k!}G(x)^{\beta k}.$$

Further applying binomial series expansion gives;

$$[S(x)]^{n-p+i}f(x) = \lambda\beta g(x)e^{\lambda(n-p+i+1)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=o}^{\infty}\sum_{m=0}^{\infty}\sum_{n=o}^{m}\frac{(-1)^{j+m}[\lambda(n-p+i+1)]^j}{i!k!} \times (j+1)^k\binom{\beta(k+1)+l}{l}\binom{\beta(k+1)+l-1}{m}\binom{m}{n}G(x)^n. \tag{4.18}$$

Subsequently, substituting the expression of $[S(x)]^{n-p+i}f(x)$ in equation (4.18) into that of $f_{X_{p:n}}(x)$ in equation (4.17) yields

$$f_{X_{p:n}}(x) = \lambda\beta\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{i+j+m}n![\lambda(n-p+i+1)]^je^{\lambda(n-p+i+1)}}{j!k!(p-1)!(n-p)!} \times (j+1)^k\binom{p-1}{i}\binom{\beta(k+1)+l}{l}\binom{\beta(k+1)+l-1}{m} \times \binom{m}{n}g(x)G(x)^n,$$

hence the proof.

4.4.3.1 Moments of Order Statistics

Proposition 4.8. The r^{th} non central moment of the p^{th} order statistic, $E(X^r_{p:n})$, of the OC family of distributions is given by,

$$E(X^r_{p:n}) = \lambda\beta\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}D_{ijklmn}\tau_{r,n}, p = 1, 2, ..., n, r = 1, 2, ..., \tag{4.19}$$

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where $\tau_{r,n} = \int\limits_{-\infty}^{\infty} x^r g(x)G(x)^n dx$ is the probability weighted moment of the baseline distribution.

Proof. The r^{th} moment of the p^{th} order statistic of a random variable is defined as

$$E(X_{p:n}^r) = \int\limits_{-\infty}^{\infty} x^r f_{X_{p:n}}(x)dx, p = 1, 2, ..., n, r = 1, 2, ...$$

Hence, substituting the expresson for the pdf of the p^{th} order statistic in equation (4.16) into the definition of $E(X_{p:n}^r)$, completes the proof.

4.4.4 Stochastic Ordering

Stochastic ordering is used to show the ordering mechanism of a dataset. A random variable X with cdf $F_X(x)$ is less than Y with cdf $F_Y(x)$ in likelihood ratio order $(X \leq_{lr} Y)$, if the function $f_X(x)/f_Y(x)$ is decreasing for all x.

Proposition 4.9. Let $X \sim OC(x; \lambda_1, \beta, \psi)$ and $Y \sim OC(x; \lambda_2, \beta, \psi)$, X is smaller than Y in stochastic order $(X \leq_{st} Y)$ if $\lambda_2 < \lambda_1$.

Proof. The ratio of the pdfs of $X \sim OC(x; \lambda_1, \beta, \psi)$ and $Y \sim OC(x; \lambda_2, \beta, \psi)$ is obtained as

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_1 - \lambda_2)\left(1 - e^{\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}\right)}.$$

Taking the differential of the logarithm of the expression yields

$$\frac{d}{dx}\left[\log\left(\frac{f(x)}{g(x)}\right)\right] = \beta(\lambda_2 - \lambda_1)g(x)G(x)^2 e^{\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}.$$

Hence, if $\lambda_2 < \lambda_1$, then, $\frac{d}{dx}\left[\log\left(\frac{f_X(x)}{f_Y(x)}\right)\right] < 0$ for all x.

Thus the proof.

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4.4.5 Inequality Measures

Several fields like insurance, econometrics and reliability studies employ Lorenz and Bonferroni curves in the study of inequality measures like income and poverty.

4.4.5.1 Lorenz Curve

Lorenz curve is defined as $\mathrm{L}_F(y) = \frac{1}{\mu}\int\limits_{-\infty}^{y} xf(x)dx$, hence for the OC family of distributions, it is obtained by substituting the mixture representation of the density in equation (4.5) into the definition of Lorenz curve $\mathrm{L}_F(y)$. The Lorenz curve for the OC family is given by

$$\mathrm{L}_F(y) = \frac{\lambda\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\nu_{ijkmq}\int\limits_{-\infty}^{y} xg(x)G^q(x)dx. \qquad (4.20)$$

Alternatively, $L_F(y)$ can be expressed in terms of quantile function by letting $G(x) = u$. This implies that $x = G^{-1}(u) = Q_G(u)$ and $g(x)dx = du$. Hence, substituting these expressions of $G(x)$ into the expression of the Lorenz curve for the OC family in equation (4.20) yields the Lorenz curve in terms of quantile function for the OC family as

$$L_F(y) = \frac{\lambda\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\nu_{ijkmq}\int\limits_{0}^{G(y)} u^q Q_G(u)du, 0 \leq u \leq 1. \qquad (4.21)$$

4.4.5.2 Bonferroni Curve

Bonferroni curve is defined as $B_F(y) = \frac{L_F(y)}{Fy}$, hence for the OC family of distributions, it is obtained by substituting the expression for the Lorenz curve $L_F(y)$ in equation (4.20) into its definition.

The Bonferroni curve for the OC family of distributions is obtained as

$$B_F(y) = \frac{\lambda\beta}{\mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{y} x g(x) G^q(x) dx. \tag{4.22}$$

4.4.6 Mean Residual Life

The mean residual life is the expected residual life or the average survival time of a component after it exceeds a specific time y. It plays a very useful role in reliability studies.

Proposition 4.10. The mean residual life of an OC random variable is given by

$$\bar{M}(y) = \frac{1}{F(y)} \left[\mu - \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{y} x g(x) G^q(x) dx \right] - y. \tag{4.23}$$

Proof. The mean residual life is defined as $\bar{M}(y) = E(x - y/x > y)$, thus

$$\bar{M}(y) = \frac{1}{F(y)} \left[\mu - \int_{-\infty}^{y} x f(x) dx \right] - y.$$

Substituting the mixture representation of the density function $f(x)$ in equation (4.5) into the definition of the mean residual life $\bar{M}(y)$ completes the proof.

4.4.7 Entropy

Entropy measures the variation or uncertainty of a random variable. It is very important especially in fields related to communications. Coding theory, with its basis hinged on efficient representation of information such as audio, video or still imagery, is one of the fields that employs the use of entropy measures (Beadle et al., 2008). Signal processing community also use entropy measures to separate the signals from multiple sources (blind deconvolution)(Lake, 2006).

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4.4.7.1 Rényi's Entropy

Rényi's entropy is a widely used measure of Guassianity in many applications such as independent component analysis for blind deconvolution. The measures of Guassianity are also used for exploratory projection pursuit in searching for non-Gaussian low-dimensional projections of high-dimensional data using projection index (Lake, 2006).

Proposition 4.11. Rényi's entropy for the OC family of distributions is given by

$$\mathrm{I}_R(\delta) = \frac{1}{1-\delta} \log \left[(\lambda\beta)^{\delta} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \eta_{ijklm} \int_{-\infty}^{\infty} g(x)^{\delta} G(x)^{m} dx \right], \delta \neq 1, \delta > 0, \tag{4.24}$$

where

$$\eta_{ijklm} = \frac{(-1)^{i+l} (\lambda\delta)^{i} (i+\delta)^{j} e^{\lambda\delta}}{i!j!} \begin{pmatrix} \beta(j+\delta)+\delta+k-1 \\ k \end{pmatrix} \times \begin{pmatrix} \beta(j+\delta)-\delta+k \\ l \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix}.$$

Proof. Let X be a random variable with pdf $f(x)$, the Rényi's entropy (Rényi, 1961) is defined as;

$$\mathrm{I}_R(\delta) = \frac{1}{1-\delta} log \left[\int_{-\infty}^{\infty} f^{\delta}(x) dx \right], \delta \neq 1, \delta > 0. \tag{4.25}$$

An expression for $f^{\delta}(x)$ is obtained by algebraically manipulating $f(x)$ in equation (4.2) as follows

$$f(x)^{\delta} = \left[\lambda\beta g(x;\psi) G(x;\psi)^{\beta-1} [1-G(x;\psi)]^{-(\beta+1)} e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)} \right]^{\delta},$$

$x > 0.$

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Expanding $f(x)^{\delta}$ using Taylor series expansion,

$$f(x)^{\delta} = (\lambda\beta)^{\delta} e^{\lambda\delta} g(x)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i}(\lambda\delta)^{i}(i+\delta)^{j}}{i!j!} G(x)^{\beta(j+\delta)-\delta}[1-G(x)]^{-[\beta(j+\delta)+\delta]}.$$

Further expanding $f(x)^{\delta}$ using binomial series expansion

$$\begin{aligned} f(x)^{\delta} =& (\lambda\beta)^{\delta} e^{\lambda\delta} g(x)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-1)^{i+l}(\lambda\delta)^{i}(i+\delta)^{j}}{i!j!} \\ &\times \binom{\beta(j+\delta)+\delta+k-1}{k}\binom{\beta(j+\delta)-\delta+k}{l}\binom{l}{m} G(x)^{m}. \end{aligned} \tag{4.26}$$

Rényi's entropy for the OC family of distributions is then obtained by substituting the expression for $f^{\delta}(x)$ in equation (4.26) into the definition of $I_R(\delta)$ in equation(4.25) as

$$\begin{aligned} I_R(\delta) =& \frac{1}{1-\delta}\log\Bigg[(\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-1)^{i+l}(\lambda\delta)^{i}(i+\delta)^{j}e^{\lambda\delta}}{i!j!} \\ &\times \binom{\beta(j+\delta)+\delta+k-1}{k}\binom{\beta(j+\delta)-\delta+k}{l}\binom{l}{m}\int_{-\infty}^{\infty} g(x)^{\delta}G(x)^{m}dx\Bigg]. \end{aligned}$$

This completes the proof.

4.4.7.2 Shannon's Entropy

The Shannon's (differential) entropy (Rényi, 1961) for a random variable X with pdf $f(x)$ is a special case of the Rényi's entropy when $\delta \uparrow 1$. A very useful property of the Shannon's entropy is that, in a set of random variables with equal variance, its maximum value is attained with a Gaussian distribution. Hence, its upper bound is

$$H(x) \leq \frac{1}{2}\log(2\pi e) + \frac{1}{2}\log\left(\sigma^{2}\right),$$

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where the right hand side is the entropy when x is normal. A population is said to be non Guassian if an estimate of the entropy for a random sample is substantially lower than this upper bound (Lake, 2006).

The Shannon's entropy is defined as

$$\eta_X = E(-\log f(x)).$$

For the OC family of distributions it is obtained by substituting the mixture representation of density $f(x)$ in equation (4.5) into the definition of the Shannon's entropy η_X. Hence, the Shannon's entropy for the OC family of distributions is given by

$$\eta_X = E\left[-\log\left(\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\nu_{ijkmq}\, g(x)G(x)^q\right)\right]. \tag{4.27}$$

4.4.7.3 Delta Entropy

The $\delta-$ entropy for a random variable X with pdf $f(x)$ is defined as

$$H(\delta) = \frac{1}{1-\delta}\log\left[1-\int_{-\infty}^{\infty} f^{\delta}(x)dx\right], \delta \neq 1, \delta > 0.$$

Hence the $\delta-$ entropy for the OC family of distributions is obtained by substituting the expression of $f^{\delta}(x)$ in equation (4.26) into the definition for $\delta-$ entropy $H(\delta)$. The $\delta-$ entropy for the OC family is given by

$$H(\delta) = \frac{1}{1-\delta}\log\left[1-(\lambda\beta)^{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\eta_{ijklm}\int_{-\infty}^{\infty} g(x)^{\delta}G(x)^{m}dx\right], \tag{4.28}$$

$$\delta \neq 1, \delta > 0.$$

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4.4.8 Stress Strength Reliability

The stress strength reliability is the probability of a component to perform without fail, an assigned task under specified conditions for a given level of stress.

Proposition 4.12. The stress strength reliliability R of the OC family is given by

$$R = 1 - \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \varphi_{ijklm} \int_{-\infty}^{\infty} g(x)G(x)^{m} dx, \tag{4.29}$$

where

$$\varphi_{ijklm} = \frac{(-1)^{i+l}(2\lambda)^{i}(i+1)^{j}e^{2\lambda}}{i!j!} \begin{pmatrix} \beta(j+1)+k \\ k \end{pmatrix} \begin{pmatrix} \beta(j+1)+k-1 \\ l \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix}.$$

Proof. Suppose $X_1 \sim (\lambda, \beta, \psi)$ is a strength random variables and $X_2 \sim (\lambda, \beta, \psi)$ is a stress random variable both from the OC family. The stress strength reliability is defined as

$$R = P(X_2 < X_1) = \int_{-\infty}^{\infty} f(x)F(x)dx = 1 - \int_{-\infty}^{\infty} f(x)S(x)dx. \tag{4.30}$$

$f(x)S(x)$ in the expression of R in equation (4.30) can be agebraically manipulated using a similar concept as that used for the mixture representation of the density $f(x)$ as follows

$$f(x)S(x) = \lambda\beta e^{2\lambda} g(x)G(x)^{\beta-1}[1-G(x)]^{-(\beta+1)} e^{\left(G(x)^{-1}-1\right)^{-\beta}} e^{-2\lambda e^{\left(G(x)^{-1}-1\right)^{-\beta}}}.$$

Expanding $f(x)S(x)$ using Taylor series expansion yields;

$$f(x)S(x) = \lambda\beta e^{2\lambda} g(x) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i}(2\lambda)^{i}(i+1)^{j}}{i!j!} G(x)^{\beta(j+1)-1}[1-G(x)]^{-[\beta(j+1)+1]}.$$

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Further expanding $f(x)S(x)$ using the binomial series expansion yields

$$\begin{aligned} f(x)S(x) =& \lambda\beta e^{2\lambda} g(x) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-1)^{i+l}(2\lambda)^i(i+1)^j}{i!j!} \\ & \times \begin{pmatrix} \beta(j+1)+k \\ k \end{pmatrix} \begin{pmatrix} \beta(j+1)+k-1 \\ l \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} G(x)^m. \end{aligned} \tag{4.31}$$

Substituting the expression for $f(x)S(x)$ in equation (4.31) obtained into the definition of R in equation (4.30) yields

$$\begin{aligned} R = 1 - & \Bigg[\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-1)^{i+l}(2\lambda)^i(i+1)^j e^{2\lambda}}{i!j!} \\ & \times \begin{pmatrix} \beta(j+1)+k \\ k \end{pmatrix} \begin{pmatrix} \beta(j+1)+k-1 \\ l \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} \int_{-\infty}^{\infty} g(x)G(x)^m dx \Bigg], \end{aligned}$$

hence the proof.

Let $G(x) = u$, $x = G^{-1}(u) = Q_G(u)$, $\frac{du}{dx} = g(x)$ and $g(x)dx = du$. The stress strength reliability of the OC family can alternatively be expressed in terms of quantile function by substituting $G(x) = u$ and $g(x)dx = du$ into the expression of R in equation (4.29) to obtain

$$R = 1 - \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \varphi_{ijklm} \int_0^1 u^m du, 0 \leq u \leq 1.$$

Simplifying the expression yields

$$R = 1 - \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \left(\frac{\varphi_{ijklm}}{m+1}\right). \tag{4.32}$$

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4.5 Parameter Estimation

The parameters of the OC family are estimated in this section using maximum likelihood estimation method.

4.5.1 Maximum Likelihood Estimation

Given a random sample, $x_1, x_2, ..., x_n$, of size n, with parameters; λ, β and ψ, from the OC family of distributions. Let $\nu = (\lambda, \beta, \psi)^T$ be a $(p \times 1)$ parameter vector, the total log-likelihood function is given by

$$\begin{aligned} \ell =& n \log \lambda\beta + (\beta - 1)\sum_{i=1}^{n} \log G(x;\psi) + \lambda \sum_{i=1}^{n} \left(1 - e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right) \\ &+ \sum_{i=1}^{n} \log g(x;\psi) - (\beta+1)\sum_{i=1}^{n} \log\left[1 - G(x;\psi)\right] \sum_{i=1}^{n} \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}. \end{aligned} \tag{4.33}$$

Partially differentiating the likelihood function yields the components of the score function $U(\upsilon) = (\partial\ell/\partial\lambda, \partial\ell/\partial\beta, \partial\ell/\partial\psi)^T$ as follows

$$\frac{d\ell}{d\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \left(1 - e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right), \tag{4.34}$$

$$\begin{aligned} \frac{d\ell}{d\beta} =& \frac{n}{\beta} + \sum_{i=1}^{n} \log G(x;\psi) + \sum_{i=1}^{n} \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta} \log\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right) \\ &- \sum_{i=1}^{n} \log\left[1 - G(x;\psi)\right] - \lambda \sum_{i=1}^{n} \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta} \log\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right) e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}} \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \frac{d\ell}{d\psi} =& \sum_{i=1}^{n} \frac{g'_k(x;\psi)}{g(x;\psi)} + (\beta - 1)\sum_{I=1}^{n} \frac{G'_k(x;\psi)}{G(x;\psi)} + (\beta+1)\sum_{i=1}^{n} \frac{G'_k(x;\psi)}{\left[1 - G(x;\psi)\right]} \\ &+ \beta \sum_{i=1}^{n} \frac{G'_K(x;\psi) G(x;\psi)^{\beta-1}}{\left[1 - G(x;\psi)\right]^{\beta+1}} - \lambda\beta \sum_{i=1}^{n} \frac{G'_K(x;\psi) G(x;\psi)^{\beta-1}}{\left[1 - G(x;\psi)\right]^{\beta+1}} e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}, \end{aligned} \tag{4.36}$$

where $g'_K(x;\psi) = \frac{dg(x;\psi)}{d\psi}$, $g''_K(x;\psi) = \frac{d^2 g(x;\psi)}{d\psi^2}$, $G'_K(x;\psi) = \frac{dG(x;\psi)}{d\psi}$ and $G''_K(x;\psi) = \frac{d^2 G(x;\psi)}{d\psi^2}$.

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The estimators of the parameters are then obtained by setting equations (4.34), (4.35) and (4.36) to zero and solving them numerically using the iterative methods such as the Newton-Raphson type algorithms. The observed information matrix $J(\upsilon)$, is required for interval estimation of the parameters. It can be estimated as $J(\upsilon) = \frac{\partial^2 \ell}{\partial i \partial j}$ for $(i, j = \lambda, \beta, \psi)$ whose elements are evaluated numerically.

4.6 Some Special Distributions

Generalization of several distributions can be made using the OC family of distributions. Three special distributions; odd Chen Burr III (OCB), odd Chen Lomax (OCL) and odd Chen Weibull (OCW) were developed in this section.

4.6.1 Odd Chen Burr III Distribution

The cdf and pdf of Burr III distribution (Burr, 1942), the baseline model, are respectively $G(x) = \left(1 + x^{-\theta}\right)^{-\gamma}$ and $g(x) = \gamma\theta x^{-\theta-1}\left(1 + x^{-\theta}\right)^{-\gamma-1}$, $x > 0, \theta > 0, \gamma > 0$. Substituting $G(x)$ and $g(x)$ into equations (4.1), (4.2) and (4.4), respectively yields the cumulative distribution, probability density and hazard functions of the OCB distribution. The cdf and pdf of the OCB distribution are respectively given by

$$F(x) = 1 - e^{\lambda\left(1 - e^{\left[\left(1+x^{-\theta}\right)^{\gamma} - 1\right]^{-\beta}}\right)}, x > 0, \theta > 0, \beta > 0, \gamma > 0, \lambda > 0 \tag{4.37}$$

and

$$\begin{aligned} f(x) =& \lambda\beta\gamma\theta x^{-(\theta+1)}\left(1 + x^{-\theta}\right)^{(-\gamma\beta+1)}\left[1 - \left(1 + x^{-\theta}\right)^{-\gamma}\right]^{-(\beta+1)} e^{\left[\left(1+x^{-\theta}\right)^{\gamma}-1\right]^{-\beta}} \\ & \times e^{\lambda\left(1 - e^{\left[\left(1+x^{-\theta}\right)^{\gamma} - 1\right]^{-\beta}}\right)}, x > 0. \end{aligned} \tag{4.38}$$

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Its hazard function is given by

$$h(x) = \lambda\beta\gamma\theta x^{-(\theta+1)}\left(1+x^{-\theta}\right)^{-(\gamma\beta+1)}\left[1-\left(1+x^{-\theta}\right)^{-\gamma}\right]^{-(\beta+1)} e^{\left[\left(1+x^{-\theta}\right)^{\gamma}-1\right]^{-\beta}}, \quad x > 0. \tag{4.39}$$

The OCB distribution exhibits; increasing, decreasing, unimodal left and right skewed shapes of density function. For some selected values, it exhibits; bathtub, upside down bathtub, modified upside down bathtub, decreasing and increasing failure rates as shown by its density and hazard rate plots in Figure 4.1.

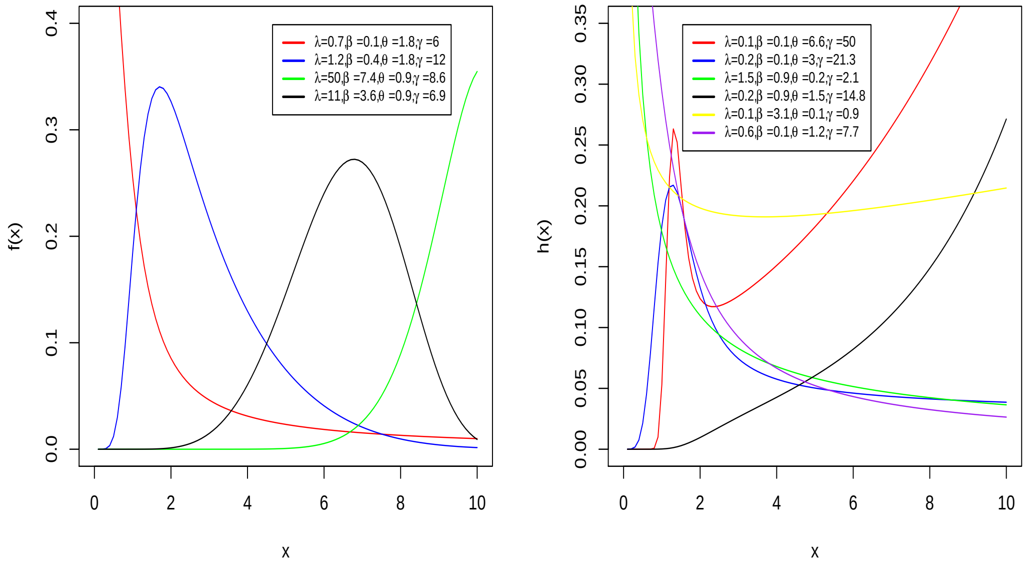

Figure 4.1: Plots of density and hazard rate functions of OCB distribution

The quantile function $Q_G(u)$ for the Odd Chen Burr III distribution is given by

$$Q_G(u) = \left\{\left[\frac{\left(\log\left(1-\left(\frac{\log(1-u)}{\lambda}\right)\right)\right)^{\frac{1}{\beta}}}{1+\left(\log\left(1-\left(\frac{\log(1-u)}{\lambda}\right)\right)\right)^{\frac{1}{\beta}}}\right]^{-\frac{1}{\gamma}} - 1\right\}^{-\frac{1}{\theta}}, 0 \le u \le 1. \tag{4.40}$$

The plots of skewness and kurtosis of the OCB distribution are shown in Figure 4.2.

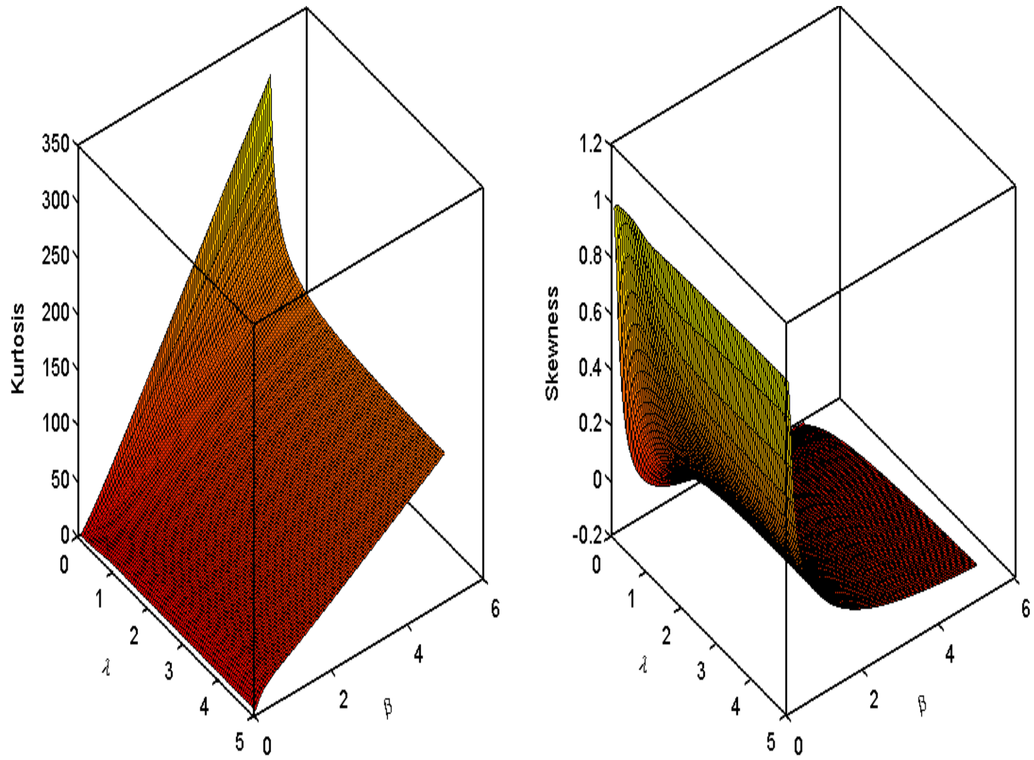

Figure 4.2: Plots of skewness and kurtosis of OCB distribution

The plots in Figure 4.2 reveals that varying combinations of the parameters have varying effects on the measures of skewness and kurtosis of the OCB distribution.

4.6.2 Odd Chen Lomax Distribution

The cdf and pdf of the Lomax distribution (Lomax, 1954) are respectively given by $G(x) = 1 - (1 + \theta x)^{-k}$ and $g(x) = \theta k(1 + \theta x)^{-(k+1)}$, $x > 0, k > 0, \theta > 0$. The cdf and pdf of the OCL distribution is obtained by substituting the cdf and pdf of the Lomax distribution into the cdf and pdf of the OC generator in equations(4.1) and (4.2) respectively. The cdf and pdf of the OCL distribution are respectively given by

$$F(x) = 1 - e^{\lambda\left(1-e^{\left[(1+\theta x)^k-1\right]^\beta}\right)}, x > 0, \lambda > 0, \theta > 0, \beta > 0, k > 0 \qquad (4.41)$$

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and

$$f(x) = \lambda\beta\theta k(1+\theta x)^{\beta k-1}\left[1-(1+\theta x)^{-k}\right]^{\beta-1} e^{\left[(1+\theta x)^{k}-1\right]^{\beta}} e^{\lambda\left(1-e^{\left[(1+\theta x)^{k}-1\right]^{\beta}}\right)},$$
$$x > 0. \tag{4.42}$$

Its hazard function is given by

$$h(x) = \lambda\beta\theta k(1+\theta x)^{\beta k-1}\left[1-(1+\theta x)^{-k}\right]^{\beta-1} e^{\left[(1+\theta x)^{k}-1\right]^{\beta}}, x > 0. \tag{4.43}$$

The density plot of the OCL distribution exhibit varying shapes such as; increasing, decreasing and non monotonically increasing shapes among others, as shown in Figure 4.3. The hazard rate function exhibited; upside down bathtub, decreasing and increasing failure rates, for some selected values.

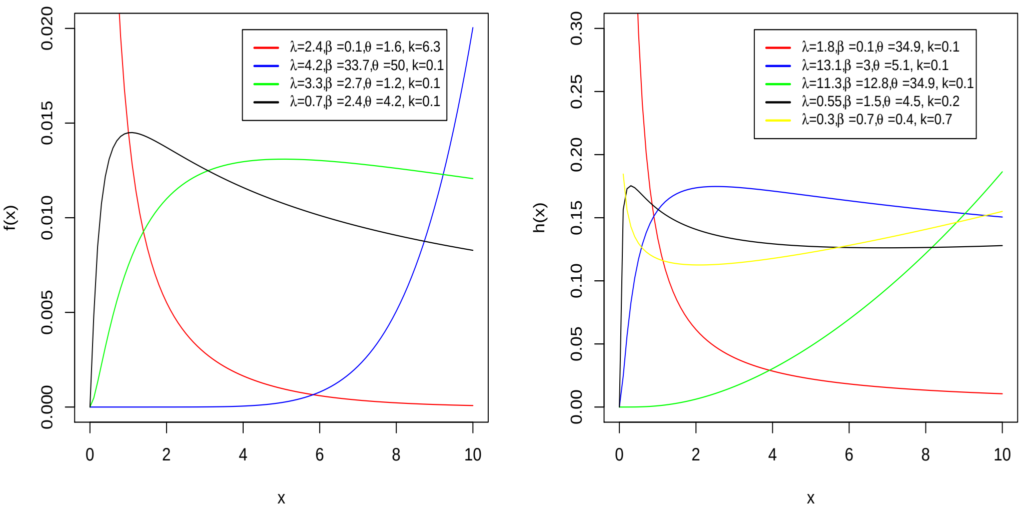

Figure 4.3: Plots of density and hazard rate functions of OCL distribution

The quantile function for the Odd Chen Lomax distribution is obtained as

$$Q_G(u) = \frac{1}{\theta}\left[\left(1-\left(\frac{\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}{1+\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}\right)\right)^{-\frac{1}{k}}-1\right], 0 \leq u \leq 1. \tag{4.44}$$

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The plots of skewness and kurtosis of the OCL distribution is shown in Figure 4.4. From the plots it can be seen that both measures are directly affected by varying measures and combinations of the parameter values.

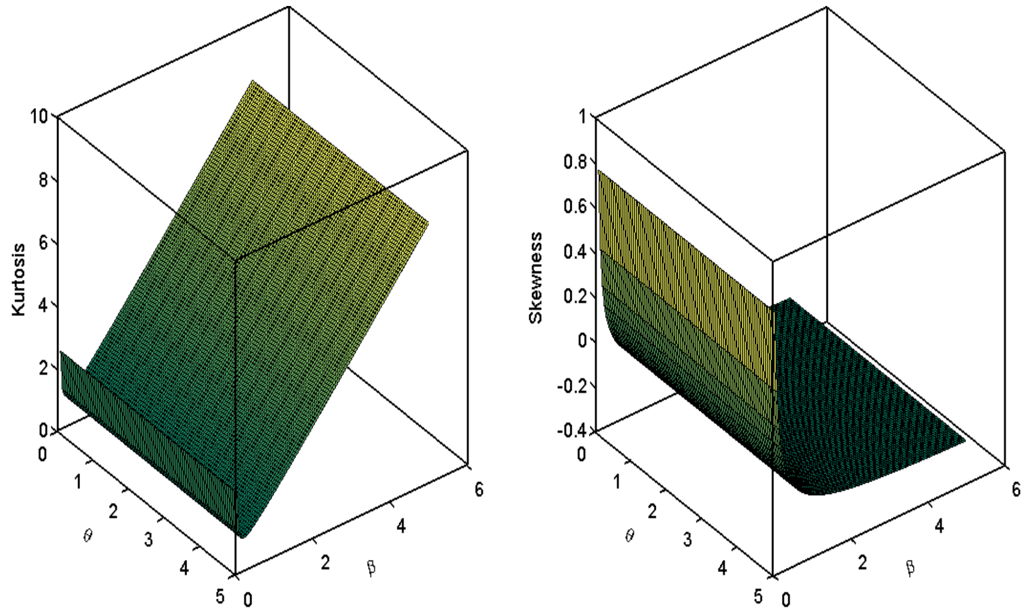

Figure 4.4: Plots of skewness and kurtosis of OCL distribution

4.6.3 Odd Chen Weibull Distribution

The Odd Chen Weibull (OCW) distribution is obtained by substituting the cdf and pdf of Weibull distribution (Weibull, 1951) given by $G(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}$ and $g(x) = \left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1} e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}$ respectively into the cdf, pdf and hazard function of the OC family given by equations (4.1), (4.2) and (4.4). The cdf, pdf and hazard function of OCW distribution are respectively obtained as

$$F(x) = 1 - \exp\left[\lambda\left(1 - \exp\left(e^{\left(\frac{x}{\alpha}\right)^{\gamma}} - 1\right)^{\beta}\right)\right], x > 0, \alpha > 0, \beta > 0, \gamma > 0, \quad (4.45)$$

$$f(x) = \lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{-\gamma\beta}} e^{\left(e^{\left(\frac{x}{\alpha}\right)^{\gamma}}-1\right)^{\beta}} e^{\lambda\left(1-\exp\left(e^{\left(\frac{x}{\alpha}\right)^{\gamma}}-1\right)^{\beta}\right)},$$
$$x > 0 \quad (4.46)$$

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and

$$h(x) = \lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma\beta}}e^{\left(e^{\left(\frac{x}{\alpha}\right)^{\gamma}}-1\right)^{\beta}}, x > 0. \qquad (4.47)$$

A display of plots of the density and hazard rate functions of the OCW distribution are found in Figure 4.5. The density plot shows shapes such as symmetric, unimodal right skewed, J and reversed J shapes. The hazard rate plot for some selected values exhibits increasing and decreasing failure rates, bathtub and upside down bathtub shapes.

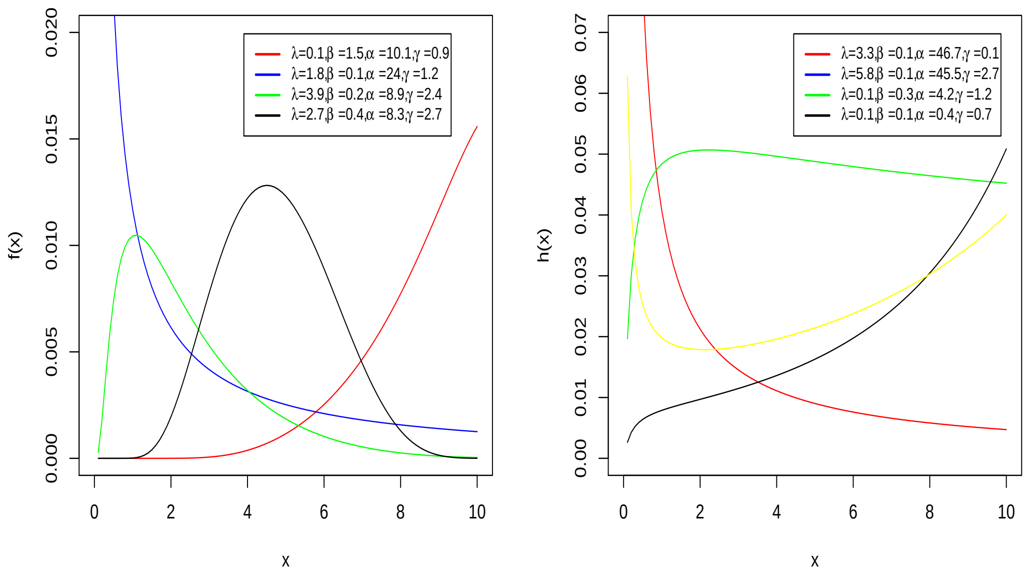

Figure 4.5: Plots of density and hazard rate functions of OCW distribution

The quantile function $Q_G(u)$ the Odd Chen Weibull distribution is given by

$$Q_G(u) = \alpha\left[-\log\left(1-\frac{\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}{1+\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}\right)\right]^{\frac{1}{\gamma}}, 0 \leq u \leq 1. \qquad (4.48)$$

The OCW distribution can model datasets exhibiting different degrees of skewness and kurtosis as shown in Figure 4.6.

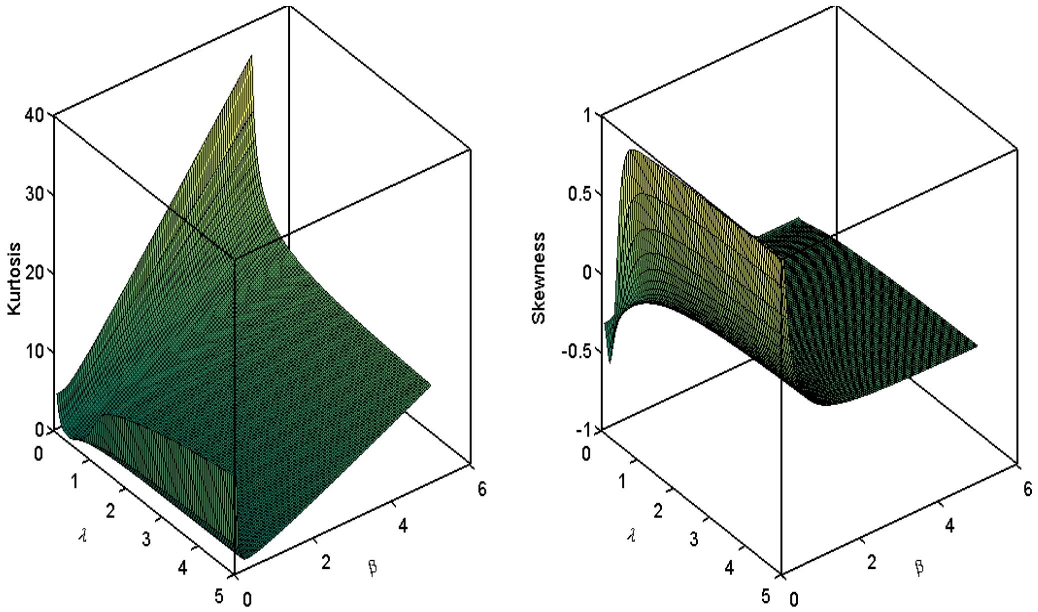

Figure 4.6: Plots of skewness and kurtosis of OCW distribution

4.7 Simulation

Validation of the maximum likelihood estimators is carried out in this section using Monte Carlo simulations. This is done using the estimators of the OCW distribution. Random numbers from the OCW distribution are generated using the OCW quantile function in equation (4.48). Setting the initial parameter values; $\theta = 0.3$, $\beta = 0.8$, $\gamma = 0.1$ and $\lambda = 0.4$, for sample sizes $n = 50, 150, 300, 600, 1000$, the simulations are repeated 1500 times for each sample. Repeating similar sample sizes for the initial parameter values;$\theta = 0.9$, $\beta = 3.5$, $\gamma = 2.5$ and $\lambda = 0.6$, the simulations are repeated for each sample another 1500 times. The root mean square error (RMSE) and the average bias (AB) for the estimators of the parameters were computed using the expressions

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2}$$

and

$$AB = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right).$$

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The RMSE, AB and coverage probability (CP) for the estimators of the parameters at 95% confidence intervals are presented in Table 4.1. From the table it is observed that there is convergence of the RMSE and AB in all cases. Thus they decrease to zero(0) as the sample size increases. The CPs are also observed to be close the nominal value of 0.95. This emphasizes the effectiveness of the method of maximum likelihood in estimating the parameters of the OCW distribution.

Table 4.1: Simulation results of AB, RMSE and CP for OCW distribution.

Parameter	n	I			II		
		AB	RMSE	CP	AB	RMSE	CP
θ	50	0.5467	0.7802	0.8880	2.8948	3.8475	0.9840
	150	0.3836	0.6492	0.8673	1.6762	2.4370	0.9560
	300	0.2496	0.5282	0.8633	1.1345	1.6802	0.9467
	600	0.1365	0.3922	0.8600	0.7915	1.1433	0.9600
	1000	0.0872	0.3048	0.8853	0.6119	0.8900	0.9540
β	50	0.0388	0.1555	0.9833	-2.1826	2.5354	0.4593
	150	0.0249	0.0812	0.9740	-1.8235	2.2538	0.5820
	300	0.0165	0.0567	0.9820	-1.6394	2.0520	0.6387
	600	0.0117	0.0371	0.9813	-1.4499	1.8822	0.6523
	1000	0.0072	0.0263	0.9727	-1.2787	1.7325	0.6440
γ	50	-0.0129	0.0962	0.9999	246.9147	966.0244	0.9767
	150	-0.0081	0.0732	0.9967	28.4697	107.9522	0.9887
	300	-0.0014	0.0620	0.9740	7.6163	21.8327	0.9953
	600	0.0021	0.0479	0.9620	2.9083	4.8612	0.9999
	1000	0.0016	0.3880	0.9553	2.0458	3.2785	0.9900
λ	50	0.1021	0.4368	0.8947	0.8223	17.0391	0.6080
	150	0.0150	0.2060	0.9267	-0.1217	0.7355	0.6553
	300	-0.0064	0.1398	0.9307	-0.1766	0.4760	0.7100
	600	-0.0104	0.0972	0.9433	-0.1953	0.3604	0.7367
	1000	-0.0084	0.0772	0.9393	-0.1898	0.3135	0.7487

4.8 Applications

In this section, real life datasets are used to demonstrate the applications of the OCB, OCL and OCW distributions in providing good parametric fit. The maximum likelihood estimates for the parameters of the model were obtained by maximizing the log-likelihood function of the models. Their performance was then compared with that of Chen distribution and new generalized Weibull(NGW) distribution (Zaindin and Sarhan, 2011) using the AD, CM and KS goodness of fit measures and the AIC, BIC and CAIC information criteria measures. The

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smaller the value of the goodness of fit measures the better the fit to the data. The negative log-likelihood was also considered for the sake of comparison. The cdf and pdf of the NGW are respectively given by $F(x) = \left[1 - e^{-\alpha x - \beta x^{\theta}}\right]^{\lambda}$ and $f(x) = \lambda \left(\alpha + \beta\theta x^{\theta-1}\right) e^{-\alpha x - \beta x^{\theta}} \left[1 - e^{-\alpha x - \beta x^{\theta}}\right]^{\lambda-1}$.

4.8.1 First Application

Data 1 which consists lifetimes of 50 components, given by Aarset (1987), was used in this application. Descriptive statistics of the dataset in Table 4.2 shows that, the minimum and maximum lifetimes of the components recorded were 0.1 and 86 respectively. The average lifetime of a component was 45.686 with a standard deviation of 32.8353. The high standard deviation might be an indication of the presence of extreme values. The values of the coefficients of skewness and kurtosis are -0.1378 and 1.4139 respectively which indicates the dataset is left-skewed and heavy-tailed (leptokurtic).

Table 4.2: Descriptive statistics of the lifetimes of 50 components

Minimum	Maximum	Mean	Standard deviation	Skewness	Kurtosis
0.1	86	45.686	32.8353	-0.1378	1.4139

A total time on test (TTT) transform plot of the dataset in Figure 4.7 shows that the dataset exhibits a modified bathtub shaped failure rate as it starts with a convex shape with an angle below the 45^0 line, follows with a concave above the 45^0 line, then another convex shape and finally a concave shape.

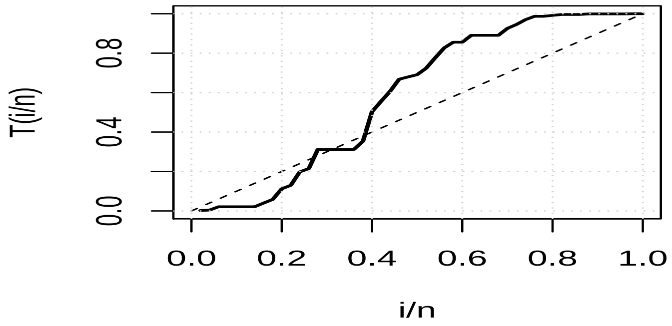

Figure 4.7: TTT-transform plot for the Data 1

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The maximum likelihood and standard error estimates of parameters for the various distributions for Data 1 are recorded in Table 4.4.

Table 4.3: Maximum likelihood and standard error estimates of parameters

Model	Parameter	Estimate	Standard error	z-value	p-value
OCB	$\hat{\lambda}$	0.0275	0.0133	2.0707	0.0384*
	$\hat{\beta}$	0.1241	0.1756	0.7068	0.4797
	$\hat{\theta}$	2.9023	3.909	0.7425	0.4578
	$\hat{\gamma}$	3.0213	1.9036	1.5872	0.1125
OCL	$\hat{\lambda}$	0.1199	0.0434	2.7591	0.0058*
	$\hat{\beta}$	0.3308	0.0828	3.9964	6.43×10^{-5}*
	$\hat{\theta}$	0.0074	0.003	2.4595	0.0139*
	$\hat{k}$	6.8406	2.2717	3.0112	0.0026*
OCW	$\hat{\lambda}$	0.3606	0.0725	4.9758	6.50×10^{-7}*
	$\hat{\beta}$	0.031	0.0071	4.3529	1.34×10^{-2}*
	$\hat{\alpha}$	45.9988	4.1288	11.1409	2.20×10^{-16}*
	$\hat{\gamma}$	5.0701	0.8432	6.0128	1.82×10^{-9}*
C	$\hat{\lambda}$	0.0205	0.0085	2.4077	0.01605*
	$\hat{\beta}$	0.3444	0.0212	16.2686	2.20×10^{-16}*
NGW	$\hat{\lambda}$	78.6862	0.0019	41180.9326	2.20×10^{-16}*
	$\hat{\beta}$	3.3297	0.1532	21.7392	2.20×10^{-16}*
	$\hat{\alpha}$	0.0245	0.0042	5.7691	7.97×10^{-9}*
	$\hat{\theta}$	0.0407	0.0138	2.9589	3.10×10^{-3}*

*: means significant at the 5% significance level

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From Table 4.3 it is observed that, all the parameters of OCL, OCW, C and NGW distributions were significant at 95% confidence level. However for the OCB distribution, only $\hat{\lambda}$ was significant at 5% level of significance, the rest of the parameter estimates ($\hat{\beta}, \hat{\theta}$ and $\hat{\gamma}$) were not significant.

OCL distribution outperforms the rest of the models as it has the highest log-likelihood and the lowest values of all the goodness-of-fit measures (KS,AD and CM) as shown in Table 4.4. It also provides a comparatively reasonable fit as illustrated by the fact that it has the lowest values for all the measures of information criteria considered (AIC, BIC and CAIC).

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Table 4.4: Log-likelihood estimates and goodness of fit measures

Model	ℓ	KS	CM	AD	AIC	BIC	CAIC
OCB	-232.01	0.168	0.318	2.001	472.025	479.673	472.914
OCL	**-225.31**	**0.145**	**0.203**	**1.439**	**458.618**	**466.266**	**459.507**
OCW	-291.65	0.191	0.255	1.606	591.291	598.940	592.180
C	-233.17	0.167	0.324	2.088	470.336	474.160	470.592
NGW	-235.60	0.162	0.380	2.368	479.209	486.857	480.098

Figure 4.8 shows the histogram of Data 1 and the densities of the fitted distributions on the left, and the empirical cdf of Data 1 and the fitted cdfs on the right.

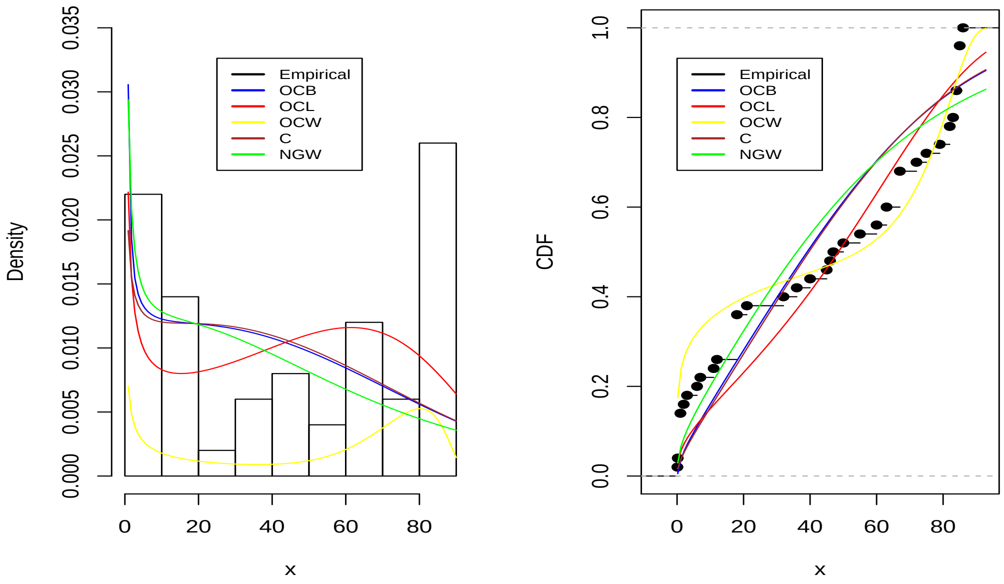

Figure 4.8: Empirical and fitted density and cdf plots of fitted distributions for Data 1

It can be observed from the plots in Figure 4.8 that the fitted distributions closely mimics the empirical density and cdf of the dataset.

The P-P plots of fitted distributions for Data 1 depicts the OCL distribution as a comparatively better fit for the dataset as shown in Figure 4.9. Though the OCB, OCW and C distributions also have their observations clustering along their diagonals, the observations for the OCL distribution are more closely clustered around the diagonal comparatively which is an indication that it provides a comparatively better fit than the rest.

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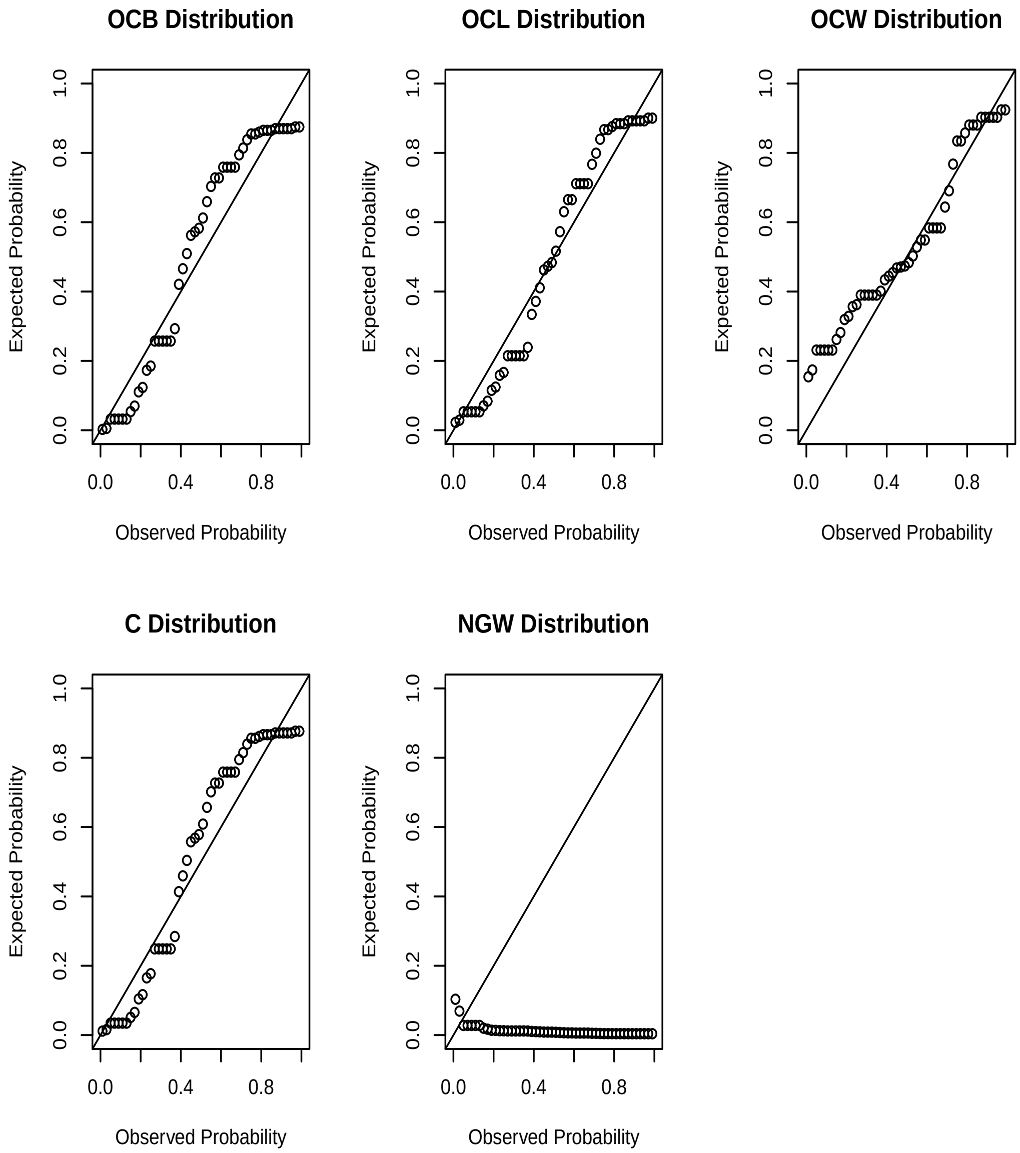

Figure 4.9: P-P plots of fitted distributions for Data 1

Figure 4.10 shows the profile log-likelihood plot of OCL distribution's parameters. From the plot its can be seen that the estimated parameter values of the OCL distribution are the maxima.

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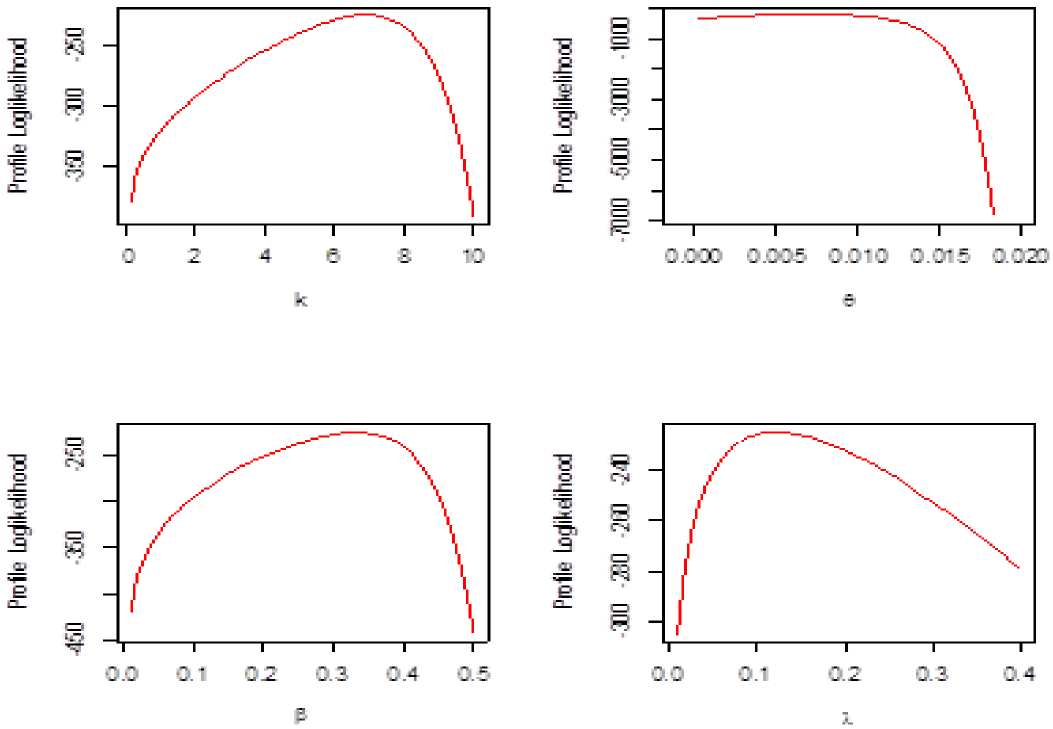

Figure 4.10: Profile log-likelihood plot of OCL parameters

4.8.2 Second Application

This application made use of Data 2 which represents the lifetime of a certain device given by Sylwia (2007). Descriptive statistics of the dataset in Table 4.2 reveals the dataset is left-skewed and thin-tailed (platykurtic) as respectively indicated by the values of the coefficients of skewness (-1.1054) and kurtosis (3.3843). The minimum and maximum lifetimes of the device recorded were 0.0094 and 12.3549. The mean lifetime of a component was 9.0395 with a standard deviation of 3.8721.

Table 4.5: Descriptive statistics of the lifetimes of a certain device

Minimum	Maximum	Mean	Standard deviation	Skewness	Kurtosis
0.0094	12.3549	9.0395	3.8721	-1.1054	3.3843

A total time on test (TTT) transform plot of the dataset in Figure 4.11 shows that the dataset exhibits a bathtub shaped failure rate as it starts with a convex shape and follows with a concave shape.

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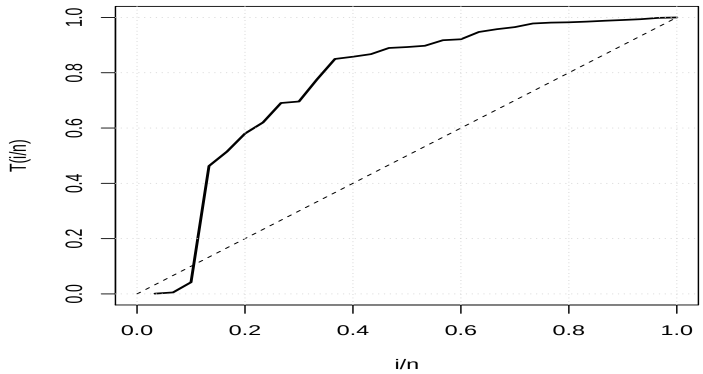

Figure 4.11: TTT-transform plot for the Data 2

The maximum likelihood and corresponding standard error estimates for Data 2 is presented in Table 4.6.

Table 4.6: Maximum likelihood and standard error estimates of parameters

Model	Parameter	Estimate	Standard error	z-value	p-value
OCB	$\hat{\lambda}$	0.0054	0.0036	1.4781	0.1394
	$\hat{\beta}$	0.0619	0.0042	14.9217	2.20×10^{-16}*
	$\hat{\theta}$	11.0047	0.0014	7737.3026	2.20×10^{-16}*
	$\hat{\gamma}$	0.4419	0.1512	2.9231	0.0035*
OCL	$\hat{\lambda}$	0.0403	0.0259	1.5534	0.1203
	$\hat{\beta}$	0.5094	0.2121	2.4016	0.0163*
	$\hat{\theta}$	0.0201	0.01	2.0149	0.0439*
	$\hat{k}$	12.2327	5.5815	2.1916	0.0284*
OCW	$\hat{\lambda}$	0.2015	0.0685	2.9409	0.0033*
	$\hat{\beta}$	0.0408	0.0181	2.2616	0.0237*
	$\hat{\alpha}$	4.7534	1.2735	3.7326	0.0002*
	$\hat{\gamma}$	3.092	0.828	3.7343	0.0002*
C	$\hat{\lambda}$	0.0064	0.0042	1.5237	0.1276
	$\hat{\beta}$	0.6886	0.0474	14.5386	2.20×10^{-16}*
NGW	$\hat{\lambda}$	106.72	0.0021	51367.5801	2.20×10^{-16}*
	$\hat{\beta}$	3.2355	0.2269	14.2622	2.20×10^{-16}*
	$\hat{\alpha}$	0.0245	0.0042	5.7691	7.97×10^{-9}*
	$\hat{\theta}$	0.0131	0.0097	1.351	0.1767

*: means significant at the 5% significance level

From Table 4.6 it can be observed that, all the parameters of OCW distribution were significant at 95% confidence level. The remaining distributions (OCB, OCL, C and NGW) had one (1) parameter each been statistically insignificant at 5% significance level.

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The OCL distribution again provided a comparatively better fit for the dataset owing to the fact that it had the highest log-likelihood and the smallest values for all the goodness-of-fit measures (KS, AD and W) as well as all information criteria (AIC, BIC and CAIC) used as shown in Table 4.7.

Table 4.7: Log-likelihood estimates and goodness-of-fit measures of fitted distributions

Model	ℓ	KS	CM	AD	AIC	BIC	CAIC
OCB	-75.97	0.1414	0.1343	0.9131	159.9333	165.5381	161.5333
OCL	**-72.19**	**0.0971**	**0.0666**	**0.4823**	**152.3888**	**157.9936**	**153.9888**
OCW	-116.5	0.0199	0.0478	0.3095	241.0092	246.614	242.6092
C	-77.24	0.1469	0.1719	1.1505	158.4785	161.2808	158.9229
NMW	-85.87	0.2219	0.4194	2.5874	179.7422	185.347	181.3422

The densities of fitted distributions and histogram of Data 2, and the empirical and fitted cdfs of the dataset are respectively shown on the left and right sides of Figure 4.12.

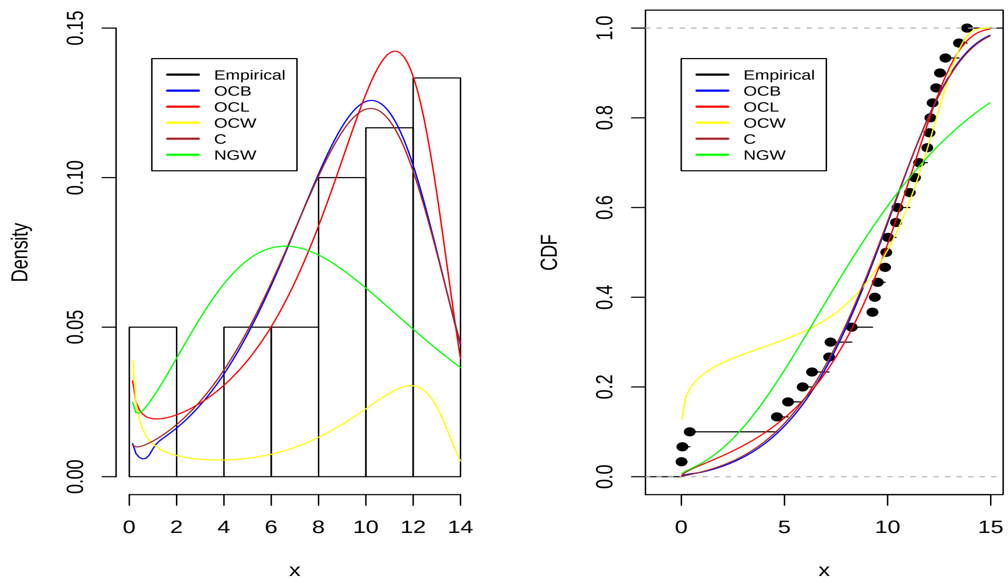

Figure 4.12: Empirical and fitted density and cdf plots of fitted distributions for Data 2

From the plots in Figure 4.12, though all the fitted distributions try to mimic the empirical density and cdf of Data 2, it is the OCL distribution that does so much closely.

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Figure 4.13 graphically confirms that the OCL provides a better fit for the dataset comparatively as its observations are more closely clustered around the diagonal than the other competing models thus OCB, OCW, C and NGW distributions.

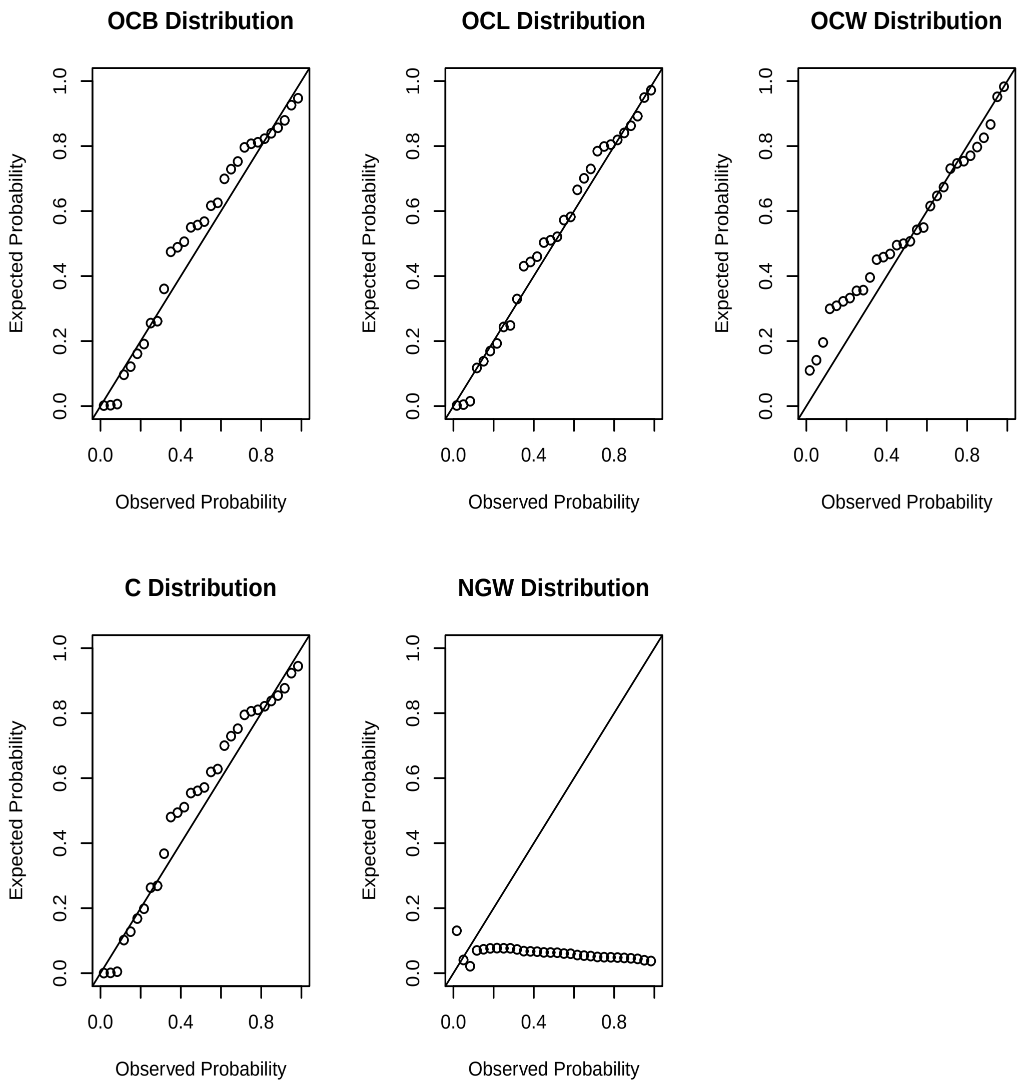

Figure 4.13: P-P plots of fitted distributions for Data 2

The profile log-likelihood plot of OCL distribution's parameters are displayed in Figure 4.14. All the estimated parameter values of the OCL distribution are the maxima as shown in the plot.

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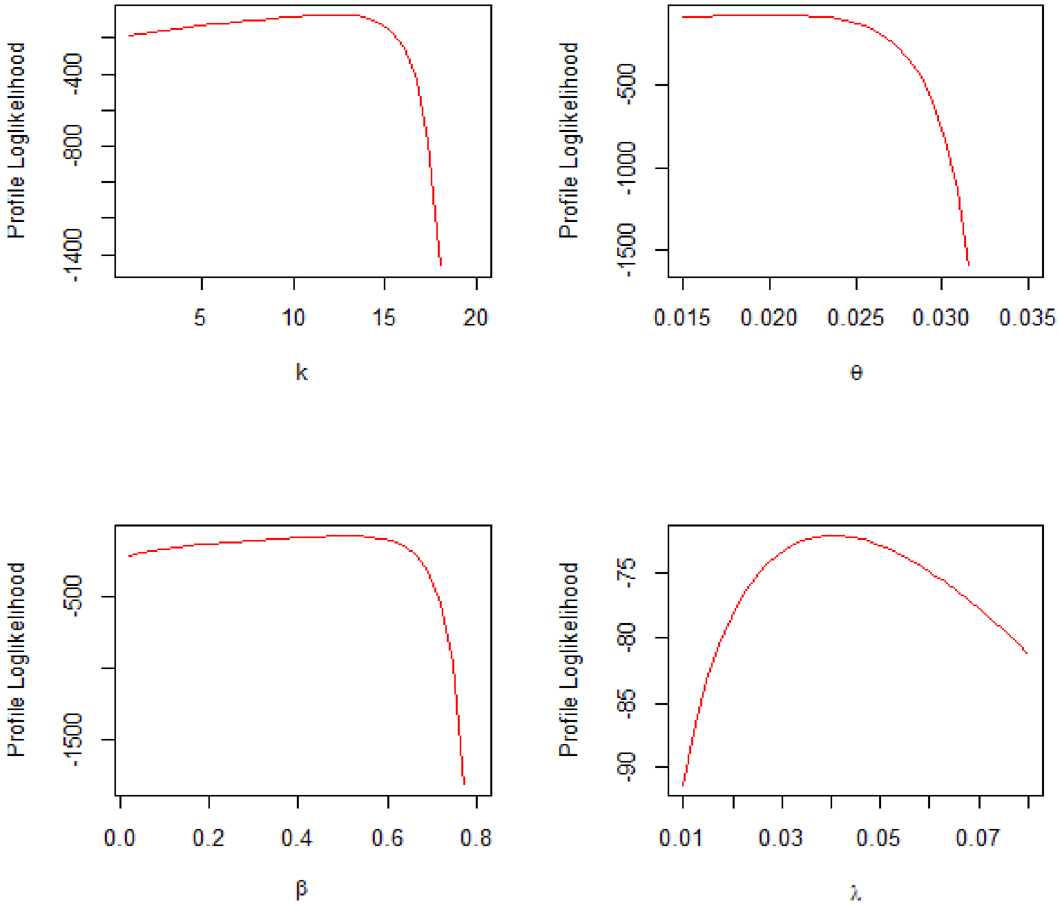

Figure 4.14: Profile log-likelihood plot of OCL parameters

4.9 Summary

The flexibility of generalized models in modeling varying datasets remains a strong motivation for developing new families of distributions. In this chapter, a new family of distribution called the OC family is developed. Statistical properties such as the stochastic ordering, order statistics, moments, uncertainty measures and entropies of the new family are derived. Maximum likelihood estimators of parameters for the family, were obtained. Three special distributions of the new family were developed and their applications demonstrated using two real datasets. A comparison of the results revealed that the OCL provided a better parametric fit to these datasets.

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CHAPTER FIVE

CHEN GENERATED FAMILY OF DISTRIBUTIONS

5.1 Introduction

The Chen generated family of distributions is presented in this chapter. Its statistical properties, such as; the quantile function, moments, stochastic orderings and order statistics among others, are derived. The parameters of the new family are estimated and new distributions developed from the new family. The application of the new models are then demonstrated using real dataset.

5.2 Chen Generated Family of Distributions

Let T be a Chen distributed continuous random variable. Suppose $G(x;\Psi)$ is the baseline cdf of an arbitrary continuous random variable X on any continuous support say $(-\infty,\infty)$ and Ψ is a $(p\times 1)$ vector of associated parameters. The cdf of the Chen generated (CG) family of distributions $(F(x))$ is defined as;

$$F(x)=\int_0^{G(x;\Psi)} f(t)dt = A\left[1-e^{\lambda\left(1-e^{G(x;\Psi)^{\beta}}\right)}\right], -\infty<x<\infty, \lambda>0, \beta>0, \tag{5.1}$$

where $A=\frac{1}{1-e^{\lambda(1-e)}}$ and λ and β are shape parameters.

The density function of the family is then obtained by differentiating the cdf in equation (5.1) as

$$f(x)=A\lambda\beta g(x;\Psi)G(x;\Psi)^{\beta-1}e^{G(x;\Psi)^{\beta}}e^{\lambda\left(1-e^{G(x;\Psi)^{\beta}}\right)}, -\infty<x<\infty. \tag{5.2}$$

Proposition 5.1. The density function of the CG family of distributions is a well-defined pdf.

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Proof. The pdf $f(x)$ of a distribution is well-defined if it is a non-negative function and when integrated over the support of X is one.

It is worth noting that $f(x)$ is non-negative. Suppose the support of X be $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} A\lambda\beta g(x;\Psi)G(x;\Psi)^{\beta-1}e^{G(x;\Psi)^{\beta}}e^{\lambda\left(1-e^{G(x;\Psi)^{\beta}}\right)}dx.$$

Let $u = \lambda\left(1 - e^{G(x;\Psi)^{\beta}}\right)$, then as $x \to -\infty$, $G(x;\Psi) \to 0$ and $u \to 0$ and as $x \to \infty$, $G(x;\Psi) \to 1$ and $u \to \lambda(1-e)$.

Also, $\frac{du}{dx} = \lambda\beta g(x;\Psi)G(x;\Psi)^{\beta-1}e^{G(x;\Psi)^{\beta}}$, implying that

$$dx = \frac{du}{\lambda\beta g(x;\Psi)G(x;\Psi)^{\beta-1}e^{G(x;\Psi)^{\beta}}}.$$

Hence

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\lambda(1-e)} Ae^{u}du = 1.$$

This completes the proof.

The survival function which is the compliment of the cdf for the CG family is given by

$$S(x) = 1 - A\left[1 - e^{\lambda\left(1-e^{G(x;\Psi)^{\beta}}\right)}\right], -\infty < x < \infty, \lambda > 0, \beta > 0, \tag{5.3}$$

whilst the corresponding failure rate function is given by

$$h(x) = \frac{A\lambda\beta g(x;\Psi)G(x;\Psi)^{\beta-1}e^{G(x;\Psi)^{\beta}}e^{\lambda\left(1-e^{G(x;\Psi)^{\beta}}\right)}}{1 - A\left[1 - e^{\lambda\left(1-e^{G(x;\Psi)^{\beta}}\right)}\right]}, -\infty < x < \infty. \tag{5.4}$$

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5.3 Mixture Representation of Distribution

Mixture representation plays a useful role in the derivation of the statistical properties of the new family of distribution. Hence the mixture representation of the pdf of the CG family of distributions is derived in this section.

Proposition 5.2. The mixture representation of the pdf of the CG family is obtained as

$$f(x) = A\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{ijkl}g(x)G(x)^{l}, -\infty < x < \infty, \tag{5.5}$$

where

$$\omega_{ijkl} = \frac{(-1)^{i+k+l}(i+1)^{j}\lambda^{i}e^{\lambda}}{i!j!}\binom{\beta(j+1)-1}{k}\binom{k}{l}.$$

Proof. By applying Taylor series expansion, the pdf of the CG family $f(x)$ in equation (5.2) is expressed as

$$f(x) = A\lambda\beta g(x)G(x)^{\beta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}\lambda^{i}}{i!}\frac{(i+1)^{j}}{j!}G(x)^{\beta(j+1)-1}.$$

The expression for $f(x)$ can be rewritten as;

$$f(x) = A\lambda\beta g(x)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}\lambda^{i}}{i!}\frac{(i+1)^{j}}{j!}[1-(1-G(x))]^{\beta(j+1)-1}.$$

Further expanding the expression for $f(x)$ using the binomial series expansion

$$(1-z)^{a-1} = \sum_{k=0}^{\infty}(-1)^{k}\binom{a-1}{k}z^{k}, |z| < 1 \tag{5.6}$$

for any real non-integer $a > 0$ yields;

$$f(x) = A\lambda\beta g(x)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\lambda^i}{i!}\frac{(i+1)^j}{j!}\sum_{k=0}^{\infty}(-1)^k\binom{\beta(j+1)-1}{k}(1-G(x))^k.$$

Assuming a in equation (5.6), an integer, the index in the sum from the expansion stops at a finite number $k = a-1$, hence

$$f(x) = A\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^{i+k+l}(i+1)^j\lambda^i e^{\lambda}}{i!j!}\binom{\beta(j+1)-1}{k}\binom{k}{l.}g(x)G(x)^l. \tag{5.7}$$

This completes the proof.

From equation (5.7), the CG family density is expressed as a product of the parameters and the sum of the product of the pdf and weighted power series of the baseline distribution.

Let $\omega^*_{ijkl} = \frac{\omega_{ijkl}}{l+1}$ and $\pi_{l+1} = (l+1)g(x)G(x)^l$. The mixture representation of the pdf $f(x)$ in equation (5.7) can alternatively be expressed in terms of the exponentiated-G (expo-G) density function as

$$f(x) = A\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega^*_{ijkl}\pi_{l+1}(x), -\infty < x < \infty, \tag{5.8}$$

where π_{l+1} is the expo-G density function with power parameter $(l+1)$.

5.4 Statistical Properties

This section discusses some of the statistical properties of the CG family of distributions. These include: quantile functions, non-central moments, moments, generating functions, inequality measures, entropies, residual life, stochastic ordering and order statistics.

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5.4.1 Quantile Function

The usefulness of the quantile function cannot be overemphasized as they are used for simulaton purposes. Also, measures of skewness and kurtosis can be computed based on the quantile measures.

Proposition 5.3. The quantile function for CG family of distributions is given by

$$Q_G(u) = x_u = G^{-1}\left(\ln\left[1 - \frac{\ln\left[(1-u/A)\right]}{\lambda}\right]\right)^{\frac{1}{\beta}}, 0 \leq u \leq 1. \tag{5.9}$$

Proof. The quantile function $Q_G(u)$ of a random variable is defined as the inverse of the cdf, $F(x_u) = P(X \leq x_u) = u, u \in (0,1)$. Replacing x with x_u in equation (5.1), equating $F(x_u)$ to u and making x_u the subject yields the quantile function. The median of the family is obtained by substituting $u = 0.5$ in equation (5.9).

5.4.2 Moments, Moment Generating Functions and Incomplete moments

The moments, moment generating functions and incomplete moments are discussed in this section.

5.4.2.1 Moments

Moments are very essential in statistical analysis as they can be used to study important features (such as tendencies, variation, skewness, kurtosis and so on) of a distribution.

Proposition 5.4. The r^{th} non-central moment of the CG family is given by

$$\mu_r^{'} = A\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{ijkl}\tau_{(r,l)}, r = 1, 2, ..., \tag{5.10}$$

where $\tau_{(r,l)} = \int_{-\infty}^{\infty} x^r g(x)(G(x))^l dx$ is the probability weighted moment of the

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baseline distribution $G(x)$.

Proof. The r^{th} non-central moment is defined as

$$E(X^r) = \mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx, r = 1, 2, ...$$

Hence, substituting the mixture form of the density $f(x)$ in equation (5.5) into the definition of $E(X^r)$, the r^{th} non-central moment of the CG family is given by

$$\mu_r^{'} = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{\infty} x^r g(x)(G(x))^l dx. \quad (5.11)$$

This completes the proof.

Let $G(x) = u$, $0 \leq u \leq 1$. This implies that $x = G^{-1}(u) = Q_G(u)$ and $g(x)dx = du$. The r^{th} non-central moment in equation (5.10), $\mu_r^{'}$ can be expressed in terms of the quantile function as

$$\mu_r^{'} = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{0}^{1} Q_G(u)^r u^l du. \quad (5.12)$$

5.4.2.2 Moment Generating Functions

Moment generating functions if they exist are very useful in establishing the moments of a random variable.

Proposition 5.5. The moment generating function of the CG family is given by

$$M_X(t) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{k} \frac{(t)^r}{r!} \omega_{ijkl}\tau_{(r,l),t=1,2,....} \quad (5.13)$$

Proof. By definition, the moment generating function is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx, t = 1, 2, ...$$

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Expanding $M_X(t)$ using Taylor series yields

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_{-\infty}^{\infty} x^r f(x) dx.$$

Substituting the expression for the r^{th} non-central moment of the CG random variable $\mu_r^{'}$ in equation (5.10) into the expression of $M_X(t)$ completes the proof.

Alternatively, letting $G(x) = u$, $0 \leq u \leq 1$, $x = Q_G(u)$ and $dx = \frac{du}{g(x)}$, the moment generating function $M_X(t)$ can be expressed in terms of quantile functions as;

$$M_X(t) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{0}^{1} e^{tQ_G(u)} u^l du. \tag{5.14}$$

5.4.2.3 Incomplete Moments

Proposition 5.6. The r^{th} incomplete moment of the CG family of distributions is given by

$$M_r(y) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} x^r g(x)(G(x))^l dx, r = 1, 2, ... \tag{5.15}$$

Proof. The r^{th} incomplete moment is defined as

$$M_r(y) = \int_{-\infty}^{y} x^r f(x) dx, r = 1, 2, ...$$

Substituting the mixture representation of the density function $f(x)$ in equation (5.5) into the definition of the incomplete moment $M_r(y)$ completes the proof.

The incomplete moments can also be expressed in terms of the quantile function. Letting $G(x) = u$, $0 \leq u \leq 1$, then $x = Q_G(u)$ and $dx = \frac{du}{g(x)}$. Substituting these terms into the expression for the incomplete moments in equation (5.15),

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the incomplete moments is expressed in terms of the quantile function as;

$$M_r(y) = A\lambda\beta \sum_{-\infty}^{y}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{ijkl}\int_{0}^{G(y)} Q_G(u)^r u^l du. \tag{5.16}$$

5.4.3 Inequality Measures

Lorenz and Bonferroni curves are applied in so many fields such as econometrics, demography, reliability, medicine and insurance. They are generally used in studying inequality measures like income and poverty.

5.4.3.1 Lorenz Curve

The Lorenz curve $L_F(y)$ for incomplete moments is defined as

$$L_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx.$$

The Lorenz curve for the CG family is obtained by substituting the mixture representation of the density $f(x)$ in equation (5.5) to yield

$$L_F(y) = \frac{A\lambda\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{ijkl}\int_{-\infty}^{y} xg(x)(G(x))^l dx. \tag{5.17}$$

The Lorenz curve for the CG family can also be expressed in terms of the quantile function by letting $G(x) = u$. Then, $x = Q_G(u)$ and $dx = \frac{du}{g(x)}$, substituting these terms into the expression for the Lorenz curve in equation (5.17) gives the Lorenz curve for the CG family in terms of the quantile function as

$$L_F(y) = \frac{A\lambda\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{ijkl}\int_{0}^{G(y)} Q_G(u)u^l du, 0 \leq u \leq 1. \tag{5.18}$$

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5.4.3.2 Bonferroni Curve

Bonferroni curve $B_F(y)$ is defined as

$$B_F(y) = \frac{L_{F(y)}}{F(y)},$$

hence that for the CG family is obtained by substituing the expression for the Lorenz curve $L_F(y)$ into equation (5.17). The Bonferroni curve for the CG family is given by

$$L_F(y) = \frac{A\lambda\beta}{\mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} x g(x)(G(x))^{l} dx. \tag{5.19}$$

5.4.4 Mean Residual Life

The mean residual life of a component (which is the average survival time of the component after it has exceeded a specific time y) is defined as $E(X - y|X > y)$.

Proposition 5.07. The mean residual life of a CG random variable is given by

$$\bar{M}(y) = \frac{1}{1 - F(y)} \left[\mu - A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} x^{r} g(x)(G(x))^{l} dx \right] - y. \tag{5.20}$$

Proof. The mean residual life is defined as

$$\bar{M}(y) = \frac{1}{1 - F(y)} \left[\mu - \int_{-\infty}^{y} x f(x) dx \right] - y.$$

The mean residual life of a CG random variable is then obtained by substituting the mixture representation of the density $f(x)$ in equation (5.5) into the definition

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of the mean residual life $\bar{M}(y)$ as follows

$$\bar{M}(y) = \frac{1}{1-F(y)}\left[\mu - A\lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{ijkl}\int_{-\infty}^{y} x^r g(x)(G(x))^l dx\right] - y,$$

hence the proof.

5.4.5 Entropy

Entropy is a measure of variation or uncertainty of a random variable. Its application spans across probability theory, engineering and science in general.

5.4.5.1 Rényi's Entropy

Proposition 5.8. Rényi's entropy for the CG random variable is given by;

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[(A\lambda\beta)^{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\varpi_{ijkl}\int_{-\infty}^{\infty} g(x)^{\delta}(G(x))^l dx\right], \delta \neq 1, \delta > 0, \tag{5.21}$$

where

$$\varpi_{ijkl} = \frac{(-1)^{i+k+l}(\lambda\delta)^i}{i!}\frac{(i+\delta)^j}{j!}e^{\lambda\delta}\binom{\beta(j+\delta)-1}{k}\binom{k}{l}.$$

Proof. The Rényi's entropy (Rényi, 1961) for a random variable with pdf $f(x)$, is defined as;

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty} f^{\delta}(x)dx\right], \delta \neq 1, \delta > 0.$$

An expression for $f^{\delta}(x)$ is obtained by algebraically manipulating $f(x)$ in equation (5.2) as follows

$$f^{\delta}(x) = (A\lambda\beta)^{\delta} g(x)^{\delta} G(x)^{\delta\beta-1} e^{\delta G(x)^{\beta}} e^{\lambda\delta} e^{-\lambda\delta e^{G(x)^{\beta}}}.$$

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Applying Taylor series expansion to $f^{\delta}(x)$ yields

$$f^{\delta}(x) = (A\lambda\beta)^{\delta} e^{\lambda\delta} g(x)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i}(\lambda\delta)^{i}(i+\delta)^{j}}{i!j!}[1-(1-G(x))]^{\beta(j+\delta)-1}.$$

Further applying binomial series expansion to the expression of $f^{\delta}(x)$ yields

$$\begin{aligned} f^{\delta}(x) =& (A\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(-1)^{i+k+l}(\lambda\delta)^{i}}{i!}\frac{(i+\delta)^{j}}{j!} \\ & \times e^{\lambda\delta} \begin{pmatrix} \beta(j+\delta)-1 \\ k \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} g(x)^{\delta}(G(x))^{l}. \end{aligned}$$

The expression for $f^{\delta}(x)$ can be rewritten as

$$f^{\delta}(x) = (A\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \varpi_{ijkl}\, g(x)^{\delta}(G(x))^{l},$$

where

$$\varpi_{ijkl} = \frac{(-1)^{i+k+l}(\lambda\delta)^{i}}{i!}\frac{(i+\delta)^{j}}{j!} e^{\lambda\delta} \begin{pmatrix} \beta(j+\delta)-1 \\ k \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}.$$

Substituting $f^{\delta}(x)$ into $I_R(\delta)$ completes the proof.

5.4.6 Stochastic Ordering

Ordering mechanism in data can easily be shown using stochastic ordering. Let X and Y be random variables with cdfs $F_X(x)$ and $F_Y(x)$ respectively. X is less than Y in likelihood ratio order ($X \leq_{lr} Y$), if the function $f(x)/g(x)$ is decreasing for all x.

Proposition 5.9. Let $X \sim CG(\lambda_1, \beta, \psi)$ and $Y \sim CG(\lambda_2, \beta, \psi)$, then X is less than Y in likelihood ratio order ($X \leq_{lr} Y$) if $\lambda_2 < \lambda_1$.

Proof. The ratio of the pdfs of X and Y is given by

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_1-\lambda_2)\left(1-e^{G(x)^\beta}\right)}.$$

To determine if the function is decreasing, the differential of the logarithm of the expression is taken as follows

$$\frac{d}{dx}\left[\log\left(\frac{f(x)}{g(x)}\right)\right] = \beta(\lambda_2-\lambda_1)g(x)G(x)^{\beta-1}e^{G(x)^\beta}.$$

It can clearly be seen from the expression that the function is decreasing for all x if $\lambda_2 < \lambda_1$.

5.4.7 Order Statistics

Proposition 5.10. The pdf for the p^{th} order statistic of the CG family of distributions is given by

$$f_{X_{p:n}}(x) = \frac{n!A\lambda\beta}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} D_{ijklm}g(x)G(x)^m, \tag{5.22}$$

where

$$D_{ijklm} = \frac{(-1)^{i+j+l+m}[\lambda(n-p+i+1)]^j}{i!}\frac{(j+1)^k}{k!} \times \binom{p-1}{i}\binom{\beta(k+1)-1}{l}\binom{l}{m}e^{\lambda(n-p+i+1)}.$$

Proof. The pdf for the p^{th} order statistic $X_{p:n}$, of an independent identically distributed random sample $X_1, X_2, ..., X_n$ of size n, $f_{X_{p:n}}(x)$, is given by;

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!}[F(x)]^{p-1}[1-F(x)]^{n-p}f(x), p = 1, 2, ..., n.$$

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Expanding $[F(x)]^{p-1}$ using binomial series expansion yields

$$[F(x)]^{p-1} = \sum_{i=0}^{p-1} (-1)^i \begin{pmatrix} p-1 \\ i \end{pmatrix} [1-F(x)]^i$$

Substituting $[F(x)]^{p-1}$ into the density of the p^{th} order statistic $f_{X_{p:n}}(x)$ yields

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{p-1} (-1)^i \begin{pmatrix} p-1 \\ i \end{pmatrix} [S(x)]^{n-p+i} f(x),$$

where $[S(x)]^{n-p+i} = [1-F(x)]^{n-p+i}$.

An expression for $[S(x)]^{n-p+i} f(x)$ can be obtained by algebraically manipulating it using a similar concept as that used for expanding the density $f(x)$ of the CG family.

Applying Taylor series expansion to $[S(x)]^{n-p+i} f(x)$ given by

$$[S(x)]^{n-p+i} f(x) = A\lambda\beta g(x) G(x)^{\beta-1} e^{G(x)^\beta} e^{\lambda(n-p+i+1)\left(1-e^{G(x)^\beta}\right)},$$

yields

$$\begin{aligned}[S(x)]^{n-p+i}\, f(x) =& A\lambda\beta g(x) G(x)^{\beta-1} e^{\lambda(n-p+i+1)} \\ &\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^j[\lambda(n-p+i+1)]^j (j+1)^k}{i!k!} G(x)^{\beta k}.\end{aligned}$$

Further applying binomial series expansion to $[S(x)]^{n-p+i} f(x)$ gives;

$$\begin{aligned}[S(x)]^{n-p+i} f(x) =& A\lambda\beta g(x) e^{\lambda(n-p+i+1)} \\ &\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-1)^{j+l+m}[\lambda(n-p+i+1)]^j (j+1)^k}{i!k!} \\ &\times \begin{pmatrix} \beta(k+1)-1 \\ l \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} G(x)^m.\end{aligned}$$

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Subsequently, substituting the expression of $[S(x)]^{n-p+i}f(x)$ into that of $f_{X_{p:n}}(x)$ yields

$$f_{X_{p:n}}(x) = \frac{n!A\lambda\beta}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{(-1)^{i+j+l+m}[\lambda(n-p+i+1)]^{j}}{i!}$$
$$\times\frac{(j+1)^{k}}{k!}\binom{p-1}{i}\binom{\beta(k+1)-1}{l}\binom{l}{m}g(x)G(x)^{m},$$

hence the proof.

5.4.7.1 Moments of Order Statistics

Proposition 5.11. The r^{th} non central moment of the p^{th} order statistic, $E(X^{r}_{p:n})$ of the CG family of distributions is given by,

$$E(X^{r}_{p:n}) = \frac{n!A\lambda\beta}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}D_{ijklm}\tau_{(r,m)} \tag{5.23}$$

where $\tau_{(r,m)} = \int_{-\infty}^{\infty} x^{r}g(x)G(x)^{m}dx$ is the probability weighted moment of the baseline distribution.

Proof. The r^{th} non-central moment of the p^{th} order statistic is given by

$$E(X^{r}_{p:n}) = \mu_{r}^{'(p:n)} = \int_{-\infty}^{\infty} x^{r}f_{X_{p:n}}(x)dx.$$

Substituting the expression for the pdf of the p^{th} order statistic $f_{X_{p:n}}(x)$ in equation (5.22) into the definition of $E(X^{r}_{p:n})$ yields

$$E(X^{r}_{p:n}) = \frac{n!A\lambda\beta}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}D_{ijklm}\int_{-\infty}^{\infty} x^{r}g(x)G(x)^{m}dx,$$

hence the proof.

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5.5 Parameter Estimation

The parameters of the CG family are estimated in this section using maximum likelihoodestimation method.

5.5.1 Maximum Likelihood Estimation

Given a random sample $x_1, x_2, ..., x_n$ of size n with parameters λ, β and ψ from the CG family of distribution. Let $\nu = (\lambda, \beta, \psi)^T$ be a $(p \times 1)$ parameter vector, the total log-likelihood function is given by

$$\begin{aligned}\ell(\nu) =& n\log A\lambda\beta + \sum_{i=1}^{n}\log g(x_i;\psi) + (\beta-1)\sum_{i=1}^{n}\log G(x_i;\psi)\\ &+\sum_{i=1}^{n}G(x_i;\psi)^{\beta} + \lambda\sum_{i=1}^{n}(1-e^{G(x_i;\psi)^{\beta}})\end{aligned} \tag{5.24}$$

The parameters are then estimated by partially differentiating the total log-likelihood function with respect to the parameters of the CG family as follows.

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \frac{n(1-e)e^{\lambda(1-e)}}{1-e^{\lambda(1-e)}} + \sum_{i=1}^{n}\Big(1-e^{G(x_i;\psi)^{\beta}}\Big), \tag{5.25}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} =& \frac{n}{\lambda} + \sum_{i=1}^{n}\log G(x_i;\psi) + \sum_{i=1}^{n}G(x_i;\psi)^{\beta}\log G(x_i;\psi)\\ &-\lambda\sum_{i=1}^{n}G(x_i;\psi)^{\beta}e^{G(x_i;\psi)^{\beta}}\log G(x_i;\psi)\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\psi} =& \sum_{i=1}^{n}\frac{g_k'(x_i;\psi)}{G(x_i;\psi)} + (\beta-1)\sum_{I=1}^{n}\frac{G_k'(x_i;\psi)}{G(x_i;\psi)} + \sum_{i=1}^{n}G_K'(x_i;\psi)G(x_i;\psi)^{\beta-1}\\ &-\lambda\beta\sum_{i=1}^{n}G_K'(x_i;\psi)G(x_i;\psi)^{\beta-1}e^{G(x_i;\psi)^{\beta}},\end{aligned} \tag{5.27}$$

where $g_K'(x_i;\psi) = \frac{\partial g(x_i;\psi)}{\partial\psi}$ and $G_K'(x_i;\psi) = \frac{\partial G(x_i;\psi)}{\partial\psi}$.

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Equating the score functions to zero and numerically solving the system of equations using techniques such as the quasi Newton-Raphson method, gives the maximum likelihood estimates for the parameters. The interval estimates of the parameters are obtained by first finding the observed information matrix given by $J(\vartheta) = \frac{\partial^2 \ell}{\partial q \partial r}$ (for $q, r = \lambda, \beta, \psi$ and $\vartheta = (\lambda, \beta, \psi)^T$), whose elements can be numerically computed. Under the regularity conditions, as $n \to \infty$, $\hat{\vartheta} \sim N_p(0, J(\hat{\vartheta})^{-1})$, where $J(\hat{\vartheta})$ is the observed information matrix evaluated at $\hat{\vartheta}$. The approximate $100(1-\rho)\%$ confidence intervals (where ρ is the significance level) can be constructed using the asymptotic normal distribution.

5.6 Some Special Distributions

The CG family of distributions can be used to extend many distributions to create more flexibility in their application. In this section some special distributions were developed.

5.6.1 Chen Burr III Distribution

Suppose that the baseline distribution is BurrIII (Burr, 1942), its cdf and pdf are given by $G(x) = \left(1 + x^{-\theta}\right)^{-\gamma}$ and $g(x) = \gamma\theta x^{-\theta-1}\left(1 + x^{-\theta}\right)^{-\gamma-1}$, $x > 0, \theta > 0, \gamma > 0$ respectively. The cdf of Chen Burr III (CB) is obtained by substituting the cdf of Burr III distribution $G(x)$ into the cdf of the CG family in equation (5.1). The cdf of CB distribution is given by

$$F(x) = A\left[1 - \exp\left(\lambda\left(1 - e^{\left(1+x^{-\theta}\right)^{-\gamma\beta}}\right)\right)\right], x > 0, \alpha > 0, \beta > 0, \gamma > 0, \lambda > 0. \tag{5.28}$$

Its corresponding density and hazard functions are respectively obtained by substituting the pdf and cdf of the Burr III distribution into the density and hazard

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functions expressions for the CG family. They are respectively given by

$$f(x) = A\lambda\beta\gamma\theta(x)^{-\theta-1}\left(1+x^{-\theta}\right)^{-\gamma\beta-1}\exp\left[\left(1+x^{-\theta}\right)^{-\gamma\beta}+\lambda\left(1-e^{\left(1+x^{-\theta}\right)^{-\gamma\beta}}\right)\right],$$
$$x > 0 \tag{5.29}$$

and

$$h(x) = \frac{A\lambda\beta\gamma\theta(x)^{-\theta-1}\left(1+x^{-\theta}\right)^{-\gamma\beta-1}\exp\left[\left(1+x^{-\theta}\right)^{-\gamma\beta}+\lambda\left(1-e^{\left(1+x^{-\theta}\right)^{-\gamma\beta}}\right)\right]}{1-A\left[1-\exp\lambda\left(1-e^{\left(1+x^{-\theta}\right)^{-\gamma\beta}}\right)\right]},$$
$$x > 0. \tag{5.30}$$

Plots of the density and hazard rate functions of the CB distribution are displayed in Figure 5.1. The density plot exhibit varying shapes including unimodal with different degrees of kurtosis, right skewed and reversed J shapes. The hazard rate function for some selected values exhibited upside down bathtub, decreasing and increasing failure rates.

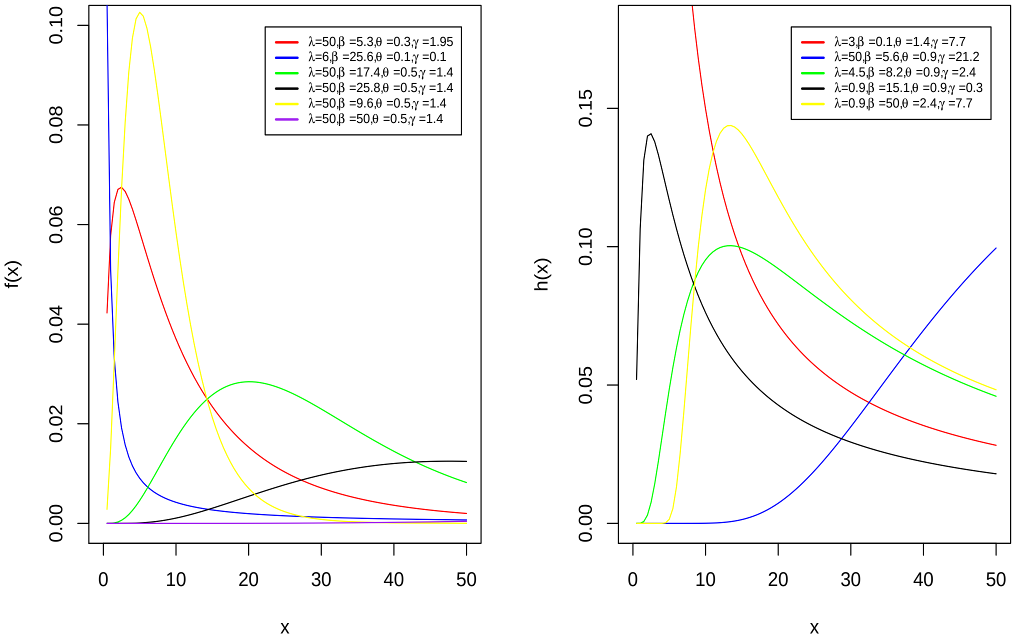

Figure 5.1: Plots of density and hazard rate functions of CB distribution

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Plots of the skewness and kurtosis of the CB distribution are shown in Figure 5.2. From the plots it can be seen that varying combinations of the parameters have varying effects on the measures of skewness and kurtosis of the CB distribution.

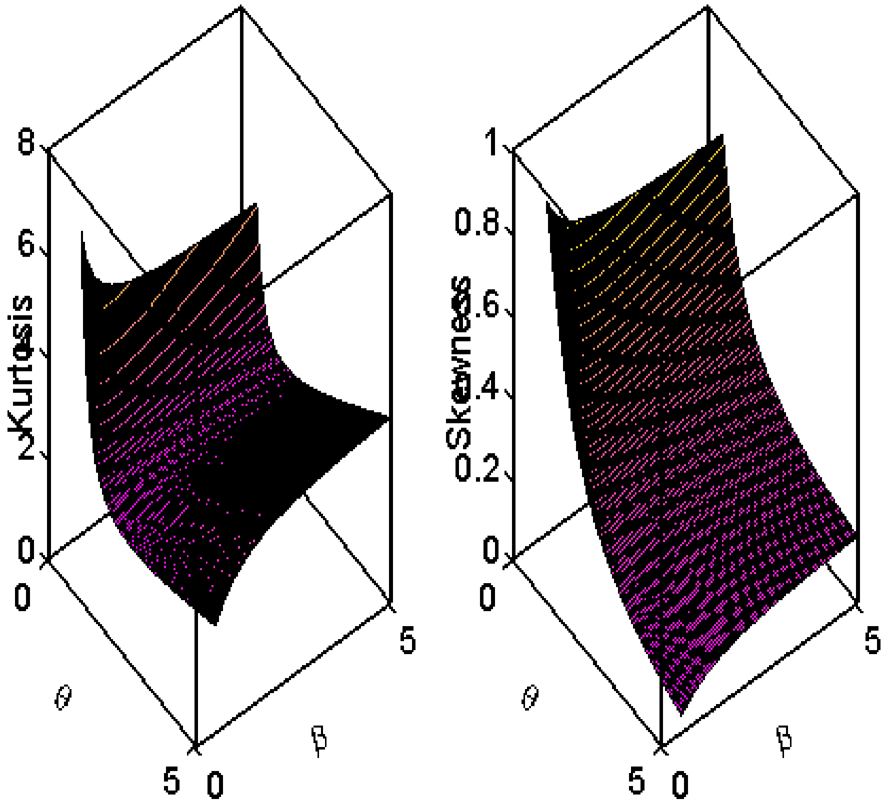

Figure 5.2: Plots of skewness and kurtosis of CB distribution

The CB distribution's quantile function is given by;

$$Q_G(u) = x_u = \left[\left(\log \left(1 - \left(\frac{\log\left(1 - {}^{u}\!/\!{}_{A}\right)}{\lambda} \right) \right) \right)^{-\frac{1}{\gamma\beta}} - 1 \right]^{-\frac{1}{\theta}}, u \in [0, 1]. \quad (5.31)$$

5.6.2 Chen Kumaraswamy Distribution

The cdf and pdf of the Kumaraswamy distribution are given by $G(x) = 1 - (1 - x^a)^b$ and $g(x) = abx^{a-1}(1 - x^a)^{b-1}$, $0 < x < 1, a > 0, b > 0$ respectively (Kumaraswamy, 1980). With Kumaraswamy distribution as the baseline distribution, the cdf, pdf and failure rate function of the Chen Kumaraswamy (CK) distribution are obtained by substituting the cdf and pdf of the Kumaraswamy distribution, thus $G(x)$ and $g(x)$ into the expressions of the cdf, pdf and failure rate function of the CG family in equations (5.1), (5.2) and (5.4) respectively.

The cdf of Chen Kumaraswamy (CK) distribution is given by

$$F(x) = A\left[1 - \exp\lambda\left[1 - e^{\left[1-(1-x^a)^b\right]^\beta}\right]\right], x > 0, a > 0, b > 0, \beta > 0, \lambda > 0, \tag{5.32}$$

with the corresponding density and hazard rate functions respectively given by

$$\begin{aligned} f(x) =& Aab\lambda\beta x^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\beta-1}\exp\left[\left(1-(1-x^a)^b\right)^\beta\right] \\ &+ \lambda\left(1-e^{\left(1-(1-x^a)^b\right)^\beta}\right), x > 0 \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} h(x) =& \frac{Aab\lambda\beta x^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\beta-1}\exp\left[\left(1-(1-x^a)^b\right)^\beta\right]}{1-\left[1-\exp\lambda\left[1-e^{\left[1-(1-x^a)^b\right]^\beta}\right]\right]} \\ &+ \frac{\lambda\left(1-e^{\left(1-(1-x^a)^b\right)^\beta}\right)}{1-\exp\lambda\left[1-e^{\left[1-(1-x^a)^b\right]^\beta}\right]}, x > 0. \end{aligned} \tag{5.34}$$

Plots of the density and hazard rate functions of the CK distribution are displayed in Figure 5.3. The plot of the density shows shapes such as the J and reversed J shapes, left and right skewed unimodal shapes, and left skewed shape. The hazard rate plot for some selected values exhibits increasing and decreasing failure rates, unimodal and bathtub shapes.

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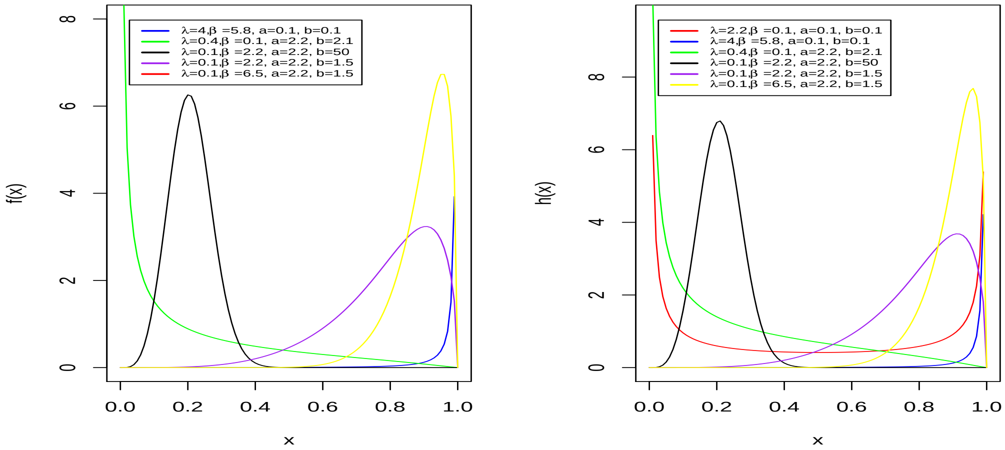

Figure 5.3: Plots of density and hazard rate functions of CK distribution

The plots of skewness and kurtosis of the CK distribution are shown in Figure 5.4. Varying measures and combinations of the parameter values have different effects on the measures of skewness and kurtosis. For instance, inceasing values of the parameter a results in increasing measures of kurtosis and a right tailed distribution as shown by the plots.

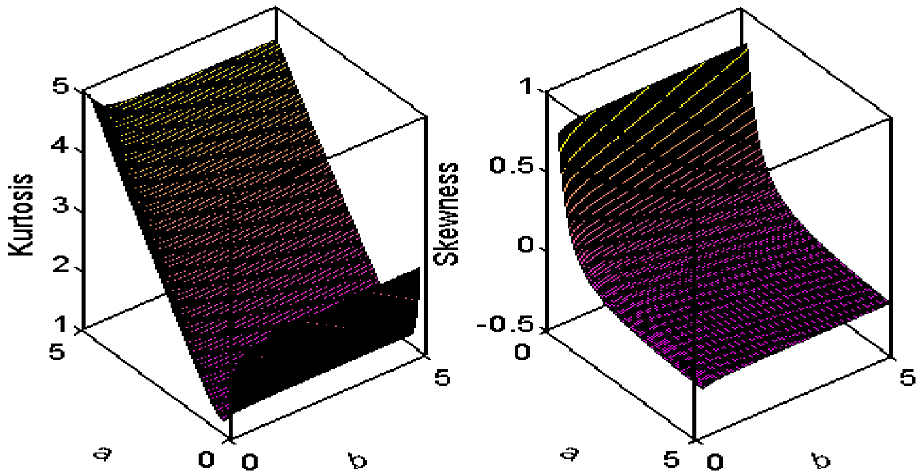

Figure 5.4: Plots of skewness and kurtosis of CK distribution

The quantile function is obtained as

$$Q_G(u) = x_u = \left[1 - \left(1 - \left(\log\left(1 - \left(\frac{\log\left(1 - {}^{u}/_{A}\right)}{\lambda}\right)\right)\right)^{\frac{1}{\beta}}\right)^{\frac{1}{b}}\right]^{\frac{1}{a}}, u \in [0,1]. \tag{5.35}$$

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5.6.3 Chen Weibull Distribution

The Chen Weibull distribution is another special ditribution developed from the CG family using the Weibull distribution (Weibull, 1951) as the baseline. The cdf and pdf of the Weibull distribution are respectively given by $G(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}$ and $g(x) = \left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1} e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}$. Substituting cdf and pdf of the Weibull distribution into the expresions of the cdf and pdf of the CG family in equations (5.1) and (5.2). The cdf and pdf of Chen Weibull (CW) distribution are respectively given by

$$F(x) = A\left[1 - \exp\lambda\left[1 - e^{\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta}}\right]\right], x > 0, \alpha > 0, \beta > 0, \gamma > 0 \tag{5.36}$$

and

$$\begin{aligned} f(x) =& A\lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta-1} \\ & \times \exp\left[\lambda\left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta} - \left(\frac{x}{\alpha}\right)^{\gamma} + \left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)\right], x > 0. \end{aligned} \tag{5.37}$$

The hazard rate function for the CG family is also obtained by substituting cdf and pdf of the Weibull distribution into the expression of the hazard rate function of the CG family in equation (5.4). The hazard rate function for the CG family is given by

$$\begin{aligned} h(x) =& \frac{A\lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta-1}}{1 - A\left[1 - \exp\lambda\left[1 - e^{\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta}}\right]\right]} \\ & \times \exp\left[\lambda\left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta} - \left(\frac{x}{\alpha}\right)^{\gamma} + \left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)\right], x > 0. \end{aligned} \tag{5.38}$$

Plots of the density exhibit right and left skewed shapes, left and right skewed unimodal shapes and reversed J shapes as shown in Figure 5.5. The hazard rate plot of the CW distribution for some selected values exhibit varying shapes such as increasing and decreasing failure rates, right and left skewed unimodal shapes

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and upside down bathtub shape.

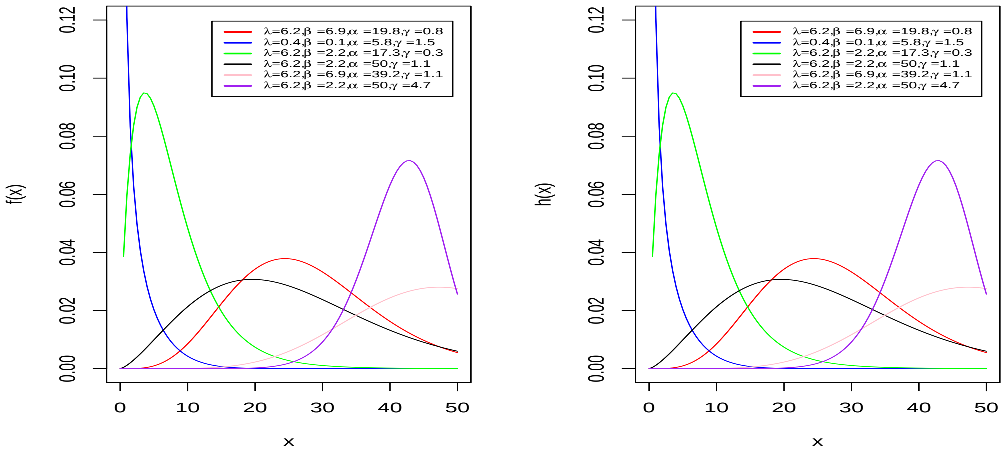

Figure 5.5: Plots of density and hazard rate functions of CW distribution

The CW distribution can model datasets exhibiting varying degrees of skewness and kurtosis as shown in Figure 5.6.

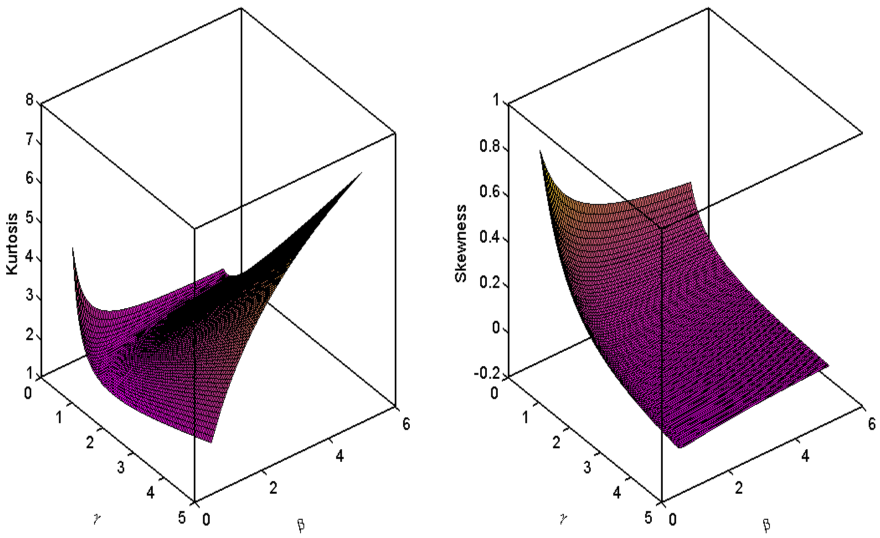

Figure 5.6: Plots of skewness and kurtosis of CW distribution

The quantile function of the CW distribution is given by

$$Q_G(u) = x_u = \alpha \left(-\log\left(1 - \left(\frac{\log\left(1 - {}^{u}\!/_{A}\right)}{\lambda}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{\gamma}}, u \in [0, 1]. \qquad (5.39)$$

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5.7 Simulations

Monte Carlo simulations were performed in this section to investigate the behavior of the maximum likelihood estimators of the parameters. For illustration purposes, the simulation experiments were undertaken using the Chen Weibull distribution. The experiments were replicated for $N = 1500$ times using sample size $n = 50, 150, 300, 600, 1000$ and parameter values $I : \lambda = 1.9, \beta = 0.9, \alpha = 0.8, \gamma = 4.8$ and $II : \lambda = 0.5, \beta = 0.5, \alpha = 0.5, \gamma = 0.5$. The average bias (AB), root mean square error (RMSE) and coverage probability (CP) of the confidence intervals for the estimators of the parameters were estimated as shown in Table 5.1.

Table 5.1: Monte Carlo Simulation Results

n	Parameter	I			II		
		AB	RMSE	CP	AB	RMSE	CP
50	λ	-0.5854	1.0708	0.9987	0.4737	0.9182	0.9913
	β	3.8663	56.4303	0.9977	2.6341	5.2212	0.9990
	α	-0.1005	0.1836	0.9977	0.6564	1.4488	0.9180
	γ	-0.0171	2.4805	0.9327	0.0876	0.303	0.9920
150	λ	-0.2607	1.1688	0.9793	0.5179	0.9948	0.9873
	β	0.6373	1.2615	0.9945	1.6927	2.3435	0.9990
	α	-0.0534	0.1321	0.9867	0.6499	1.268	0.9600
	γ	-0.1023	1.7291	0.9360	0.0652	0.1958	0.9927
300	λ	-0.1324	1.2618	0.9607	0.5254	1.0150	0.9793
	β	0.4988	0.9901	0.9973	1.5134	1.8010	0.9067
	α	-0.0396	0.1125	0.9853	0.5978	1.1114	0.9727
	γ	-0.2452	1.2307	0.9393	0.0484	0.1303	0.9913
600	λ	-0.0231	1.191	0.9592	0.4924	1.0072	0.9580
	β	0.3936	0.5929	0.9900	1.4374	1.5874	0.7500
	α	-0.0240	0.0950	0.9633	0.5487	1.0468	0.9793
	γ	-0.2420	0.9657	0.9433	0.0405	0.1034	0.9827
1000	λ	0.0428	1.1763	0.9367	0.4089	0.8572	0.9393
	β	0.3599	0.5053	0.964	1.3880	1.4766	0.6780
	α	-0.0173	0.0856	0.9407	0.4867	0.9565	0.9747
	γ	-0.2526	0.8181	0.9367	0.0402	0.0934	0.9513

From Table 5.1, the ABs and RMSEs for the estimators generally decreases to zero as the sample size increases. This implies that as the sample size increases the accuracy and consistency of the maximum likelihood estimators are achieved. Also, the CPs for most of the estimators are quite close to the nominal value of

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0.95. Thus, we can say that the maximum likelihood technique works very well to estimate the parameters of the Chen Weibull distribution.

5.8 Applications

In this section the performance of some of the new distributions developed in providing good parametric fits to real life datasets is demonstrated. Its goodness-of-fit measures are compared with competing models such as; exponentiated Chen (EC) (Chaubey and Zhang, 2015), Generalized Weibull (EW) (Mudholkar and Srivastava, 1993) and Kumaraswamy exponentiated Chen (KEC) (Khan et al., 2018) distributions.

The cdf and pdf of the EC distribution are respectively given by

$$F(x) = \left(1 - \exp\left[\lambda\left(1 - \exp\left(x^{\beta}\right)\right)\right]\right)^{\alpha}$$

and

$$f(x) = \alpha\beta\lambda x^{(\beta-1)} e^{x^{\beta}} \exp\left[\lambda\left(1 - e^{x^{\beta}}\right)\right]\left(1 - \exp\left[\lambda\left(1 - e^{x^{\beta}}\right)\right]\right)^{\alpha-1}.$$

That for EW distribution are given by

$$F(x) = \left(1 - e^{-(\lambda x)^{\gamma}}\right)^{\alpha}$$

and

$$f(x) = \alpha\gamma\lambda^{\gamma} x^{(\gamma-1)}\left(1 - e^{-(\lambda x)^{\gamma}}\right)^{\alpha-1} e^{-(\lambda x)^{\gamma}}.$$

The cdf and pdf for the KEC distribution are given by

$$F(x) = 1 - \left[1 - \left(1 - \exp\left(\alpha\left(1 - \exp(x^{\beta})\right)\right)\right)^{a\theta}\right]^{b}$$

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and

$$f(x) = \frac{ab\alpha\beta\theta x^{(\beta-1)}\exp\left(x^{\beta} + \alpha\left(1-\exp(x^{\beta})\right)\right)\left(1-\exp\left(\alpha\left(1-\exp(x^{\beta})\right)\right)\right)^{(a\theta-1)}}{\left(1-\left(1-\exp\left(\alpha\left(1-\exp(x^{\beta})\right)\right)\right)^{a\theta}\right)^{(1-b)}}.$$

In obtaining the maximum likelihood estimates for the parameters, the log-likelihood function of the models were maximized using the bbmle package's subroutine mle2 in R (Bolker, 2014). The maximum likelihood estimates with the largest maxima was chosen after using a wide range of initial values.

5.8.1 First Application

The dataset used in this application (Data 3) consists the fatigue times of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second found in Birnbaum and Saunders (1969). The minimum and maximum fatigue times of 6061-T6 aluminum coupons were 70 and 212 seconds respectively as shown by the table of descriptive statistics of the dataset in Table 5.5. The mean fatigue times of 6061-T6 aluminum coupons recorded was 133.7327 with a standard deviation of 32.8353. The dataset is positively skewed and light-tailed (platykurtic) as shown by the values of the coefficients of skewness and kurtosis which are 0.3305 and 1.0528 respectively.

Table 5.2: Descriptive statistics of the fatigue time of 6061-T6 aluminum coupons

Minimum	Maximum	Mean	Standard deviation	Skewness	Kurtosis
70	212	133.7327	22.3557	0.3305	1.0528

A preliminary exploration of the dataset on the shapes of the hazard rate function showed that the dataset has an increasing hazard rate as shown by the TTT plot in Figure 5.7 which has a concave shape.

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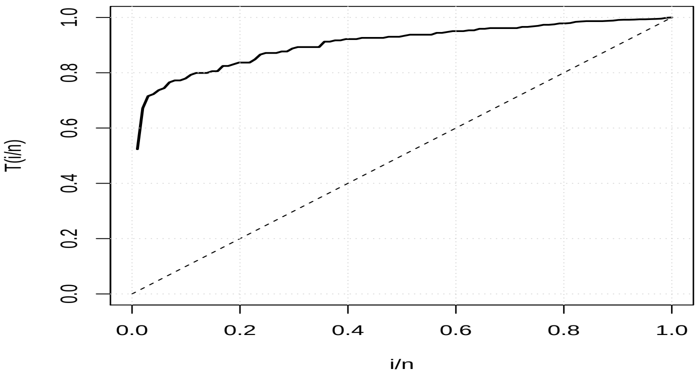

Figure 5.7: TTT-transform plot for the datasets

The maximum likelihood estimates of the parameters for the fitted distributions, and their corresponding standard errors of for the dataset are displayed in Table 5.3. All except one parameter (λ and β respectively) each of the CW and KEC distributions were significant at 5% significance level. All the other distributions (EC, EW and CB) had all their parameters significant at 5% significance level.

Table 5.3: Maximum likelihood and standard error estimates of parameters

Model	Parameter	Estimate	Standard error	z-value	p-value
CW	$\hat{\lambda}$	6.7749	4.5924	1.4752	0.1401
	$\hat{\beta}$	35.2091	2.8166	12.5006	$2.2 \times 10^{-16}*$
	$\hat{\alpha}$	49.2897	2.5753	19.1394	$2.2 \times 10^{-16}*$
	$\hat{\gamma}$	1.0187	0.1675	6.0829	$1.18 \times 10^{-9}*$
KEC	$\hat{a}$	4.3355	0.6977	6.2144	$5.151 \times 10^{-10}*$
	$\hat{b}$	2.2857	1.5572	1.4679	0.1421
	$\hat{\alpha}$	0.0209	0.0047	4.3969	$1.098 \times 10^{-5}*$
	$\hat{\beta}$	0.3237	0.0145	22.3788	$2.2 \times 10^{-16}*$
	$\hat{\theta}$	4.4725	0.6761	6.6154	$3.705 \times 10^{-11}*$
EC	$\hat{\alpha}$	1236.1	4.1476×10^{-6}	2.9802×10^{8}	$2.2 \times 10^{-16}*$
	$\hat{\beta}$	0.2446	0.0083	29.328	$2.2 \times 10^{-16}*$
	$\hat{\lambda}$	0.2889	0.0390	7.4014	$1.347 \times 10^{-13}*$
EW	$\hat{\alpha}$	55.1400	0.0007	75801.4070	$2.20 \times 10^{-16}*$
	$\hat{\beta}$	1.4931	0.1063	14.0420	$2.20 \times 10^{-16}*$
	$\hat{\lambda}$	0.0205	0.0014	14.2460	$2.20 \times 10^{-16}*$
CB	$\hat{\lambda}$	82.3588	11.9649	6.8834	$5.845 \times 10^{-12}*$
	$\hat{\beta}$	65.5650	11.9498	5.4867	$4.095 \times 10^{-8}*$
	$\hat{\theta}$	1.4247	0.0568	25.0785	$2.2 \times 10^{-16}*$
	$\hat{\gamma}$	77.4226	10.1125	7.6562	$1.916 \times 10^{-14}*$

*: means significant at the 5% significance level

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The CW distribution provides a comparatively better fit for the dataset than the KEC, EW, EC and CB distributions the as it has the lowest values for all the measures of information criteria considered (AIC, BIC and CAIC) as shown in Table 5.4. It also has the highest log-likelihood and the lowest values of all the goodness of fit measures (KS, AD and W) which is a good indication that the data follows the specified distribution.

Table 5.4: Goodness-of-fit statistics and information criteria

Model	ℓ	KS	CM	AD	AIC	BIC	CAIC
CW	**-446.99**	**0.064**	**0.046**	**0.299**	**901.984**	**912.365**	**902.41**
KEC	-447.22	0.061	0.047	0.323	904.436	917.411	905.081
EW	-450.62	0.096	0.134	0.783	907.241	915.026	907.494
EC	-446.99	0.110	0.194	1.127	913.190	920.975	913.442
CB	-456.27	0.065	0.054	0.345	920.539	931.000	920.956

This is further confirmed by the histogram of Data 3 and the densities of the fitted distributions, and the empirical and fitted cdfs for Data 3 as respecively shown on the left and right sides of Figure 5.8. From the fitted plot, it is observed that the CW distribution's density and cdf mimic the empirical density and cdf of Data 3 much more closely compared to the rest of the fitted distributions.

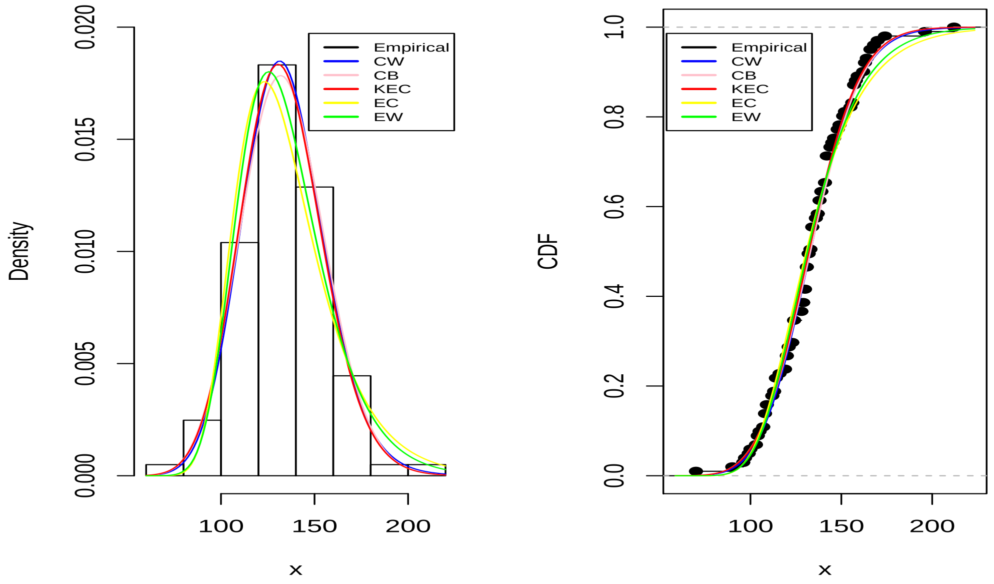

Figure 5.8: Empirical and fitted density and cdf plots of data1

The P-P plots in Figure 5.9 graphically indicates that, the CW distribution provides a better fit for the dataset in comparison with KEC, EC, EW and CB distributions, as as its observations are comparatively more closely clustered around the diagonal than the other distributions.

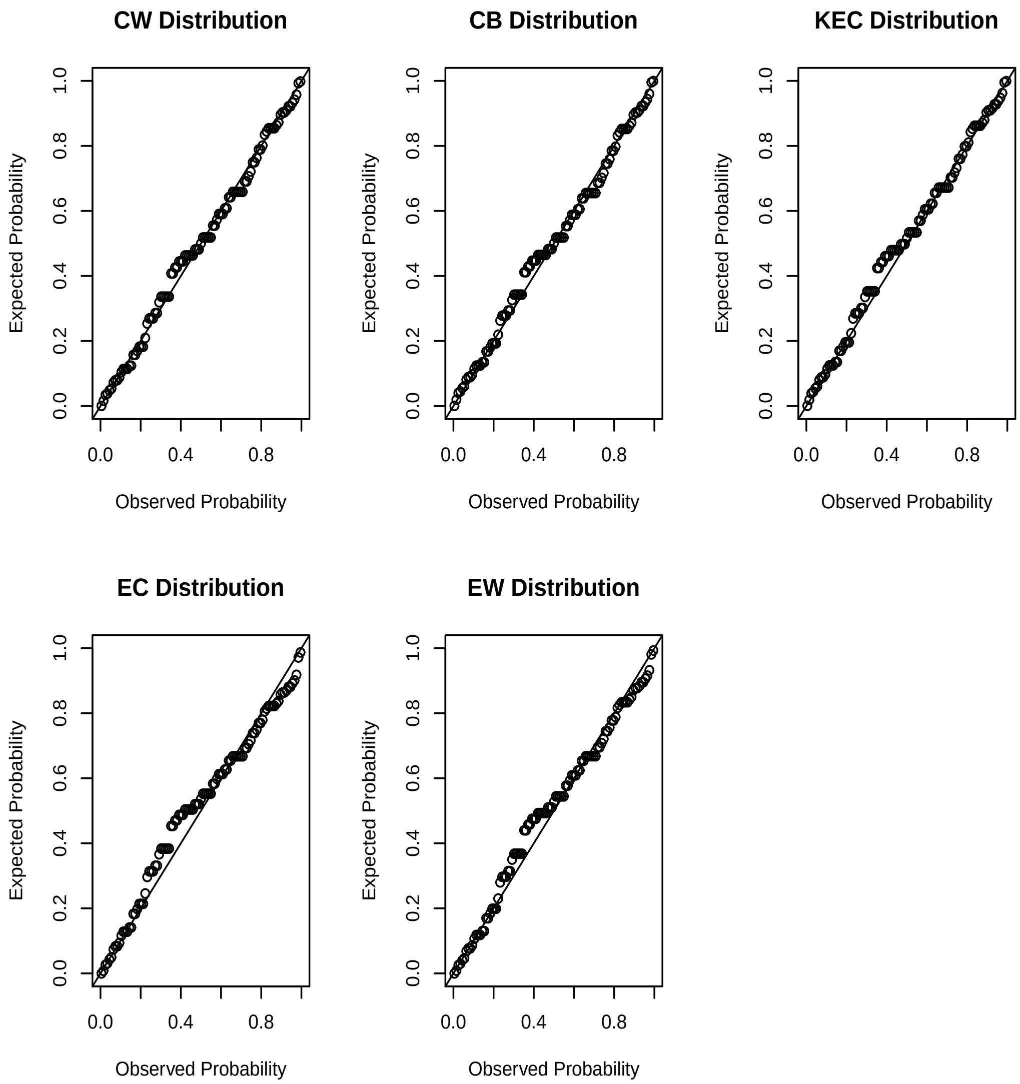

Figure 5.9: P-P plots of fitted distributions for data1

The profile likelihoods of the estimated parameters of the CW distribution for the dataset are shown in Figures 5.10 . From the plots, it is observed that all the estimated values for the parameters are the maxima.

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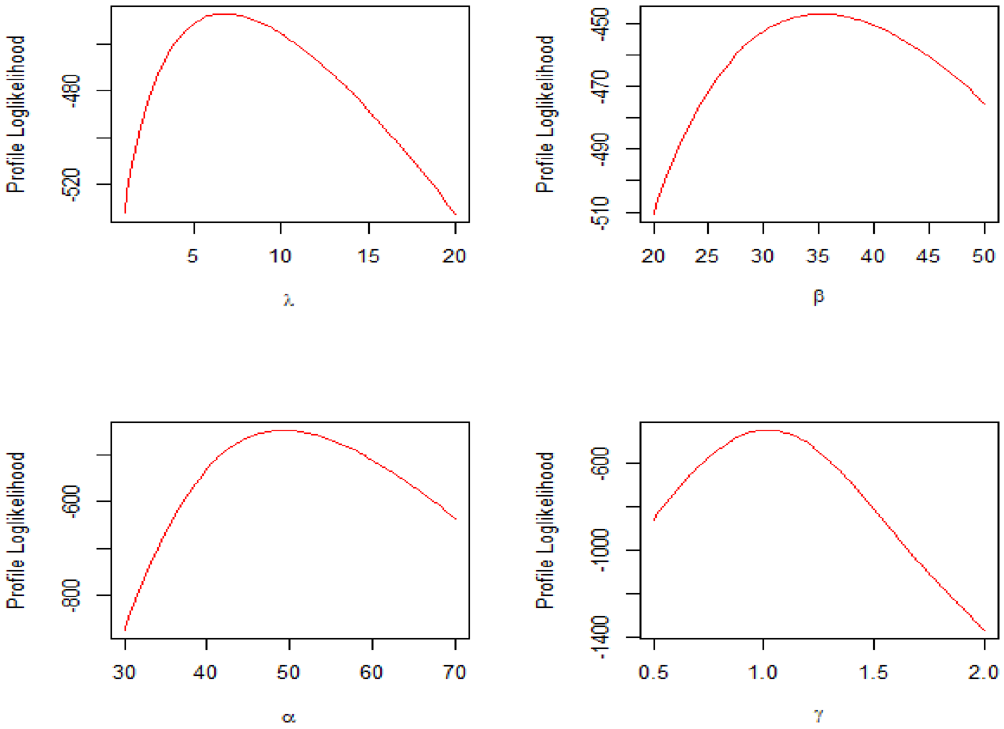

Figure 5.10: Profile log-likelihood plot of CW parameters for data1

5.8.2 Second Application

This application was carried out using Data 4 which consists the survival times (in days) of 72 guinea pigs injected with different amount of virulent tubercle bacilli studied by Bjerkedal (1960). A TTT plot of the dataset showed that the dataset has an increasing hazard rate as shown in Figure 5.11.

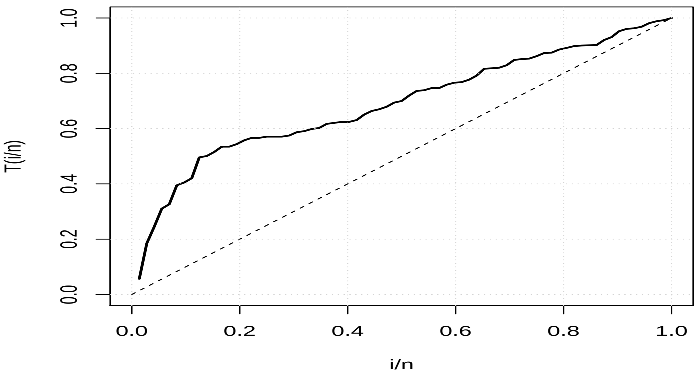

Figure 5.11: TTT-transform plot for the datasets

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Descriptive statistics of the dataset in Table 5.5 shows that the minimum and maximum survival times of the 72 guinea pigs injected with different amount of virulent tubercle bacilli were 10 and 555 days respectively. The average survival times of guinea pigs was 177 days with a standard deviation of 103 days. The high value of standard deviation might be an indication of the presence of extreme values in the dataset. The dataset is positively skewed with a coefficient of skewness value of 1.3413 and light-tailed (platykurtic) with a coefficient of kurtosis value of 1.9885.

Table 5.5: Descriptive statistics of the Survival times of guinea pigs

Minimum	Maximum	Mean	Standard deviation	Skewness	Kurtosis
10	555	176.8333	103.4654	1.3413	1.9885

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Compared to the competing models, the CW distribution with its four parameters provides a better fit for the datasets as it has the highest log-likelihood and the smallest value for all the goodness of fit measures used as shown in Table 5.6. It also has the lowest values of all the information criteria (AIC, BIC and CAIC) considered.

Table 5.6: Goodness-of-fit statistics and information criteria

Model	ℓ	KS	CM	AD	AIC	BIC	CAIC
CW	**-425.85**	**0.092**	**0.090**	**0.564**	**859.704**	**868.810**	**860.301**
CB	-426.46	0.106	0.087	0.572	860.835	869.941	861.432
KEC	-425.87	0.093	0.094	0.581	861.604	872.987	862.513
EW	-435.56	0.145	0.232	1.603	877.894	884.724	878.247
EC	-429.69	0.124	0.153	1.089	869.634	876.464	869.987

All but one parameter (λ) each of the CW and EW distributions were significant at 5% significance level as shown in Table 5.7. The rest of the distributions (EC, KEC and CB) had all their estimated parameters significant at 5% significance level.

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Table 5.7: Maximum likelihood and standard error estimates of parameters

Model	Parameter	Estimate	Standard error	z-value	p-value
CW	$\hat{\lambda}$	19.3660	36.0090	0.5378	0.5907
	$\hat{\beta}$	15.7420	1.2421	12.6736	2.2×10^{-16}*
	$\hat{\alpha}$	30.9452	12.4911	2.4774	0.0132*
	$\hat{\gamma}$	0.3095	0.0981	3.1552	0.0016*
CB	$\hat{\lambda}$	116.7489	0.1776	657.4517	2.2×10^{-16}*
	$\hat{\beta}$	1.1130	0.3878	2.8703	0.0041*
	$\hat{\theta}$	0.3893	0.0284	13.7226	2.2×10^{-16}*
	$\hat{\gamma}$	34.9340	12.1659	2.8715	0.0041*
KEC	$\hat{a}$	0.1923	0.2340	0.8217	0.4112
	$\hat{b}$	15.9390	0.0058	2761.7	2.2×10^{-16}*
	$\hat{\alpha}$	0.4490	0.4124	1.0887	0.2763*
	$\hat{\beta}$	0.1161	0.0375	3.0907	0.0020*
	$\hat{\theta}$	149.57	0.0003	521180	2.2×10^{-16}*
EC	$\hat{\alpha}$	163.36	0.0002	910695.135	2.2×10^{-16}*
	$\hat{\beta}$	0.1381	0.0079	17.481	2.2×10^{-16}*
	$\hat{\lambda}$	0.8646	0.0772	11.199	2.20×10^{-16}*
EW	$\hat{\alpha}$	242.2554	0.0941	2574.0359	2.20×10^{-16}*
	$\hat{\beta}$	0.2718	0.0220	12.3433	2.20×10^{-16}*
	$\hat{\lambda}$	4.6262	2.3780	1.9454	0.0517

*: means significant at the 5% significance level

The left side of Figure 5.12 shows the densities of the fitted distributions and a histogram of Data 4, whilst the right side displays the empirical and fitted cdfs of the dataset.

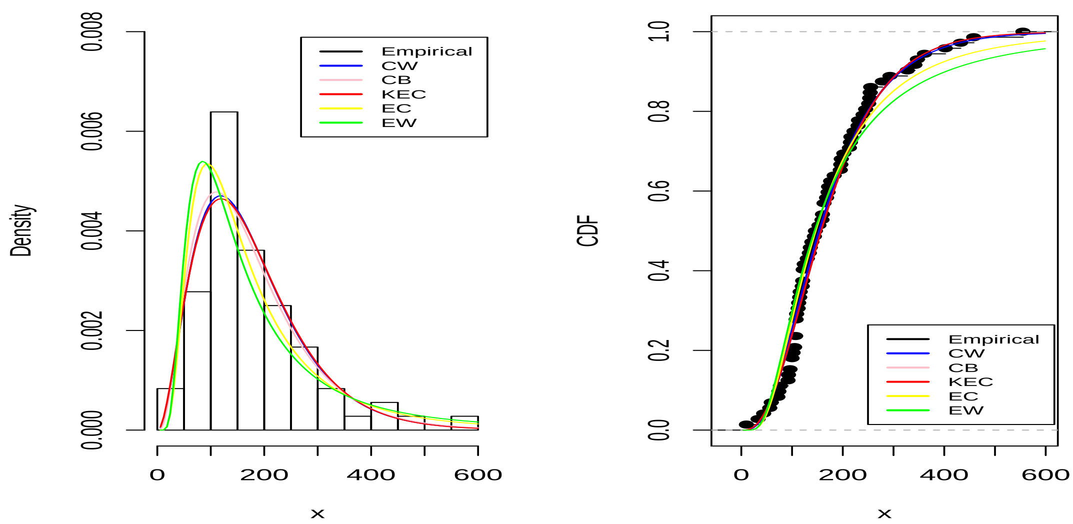

Figure 5.12: Empirical and fitted density and cdf plots of Data 4

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As seen from the plots in Figure 5.12, though all the fitted distributions mimics the empirical density and cdf of the dataset, the CW distribution does so much more closely. This further confirms that, the CW provides a comparatively reasonable fit to the dataset.

The P-P plots in Figure 5.13 also indicates the CW distribution provides a better fit for the dataset in comparison with CB, KEC, EC and EW distributions. From the plots, it can be seen that the CW distribution had its observations much more clustered along the diagonals comparatively.

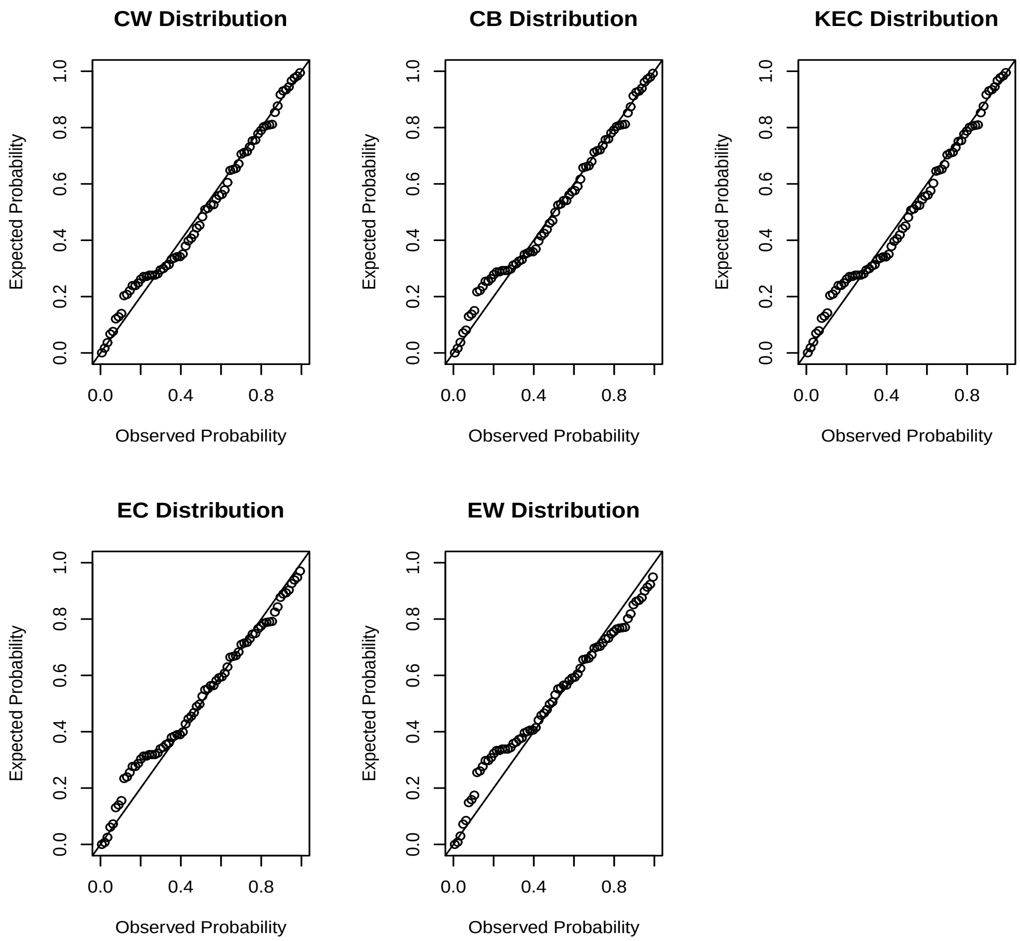

Figure 5.13: P-P plots of fitted distributions for data2

The profile likelihoods of the estimated parameters of the CW distribution for the dataset are shown in Figure 5.14. From the plots, it is observed that the estimated values for the parameters are the maxima.

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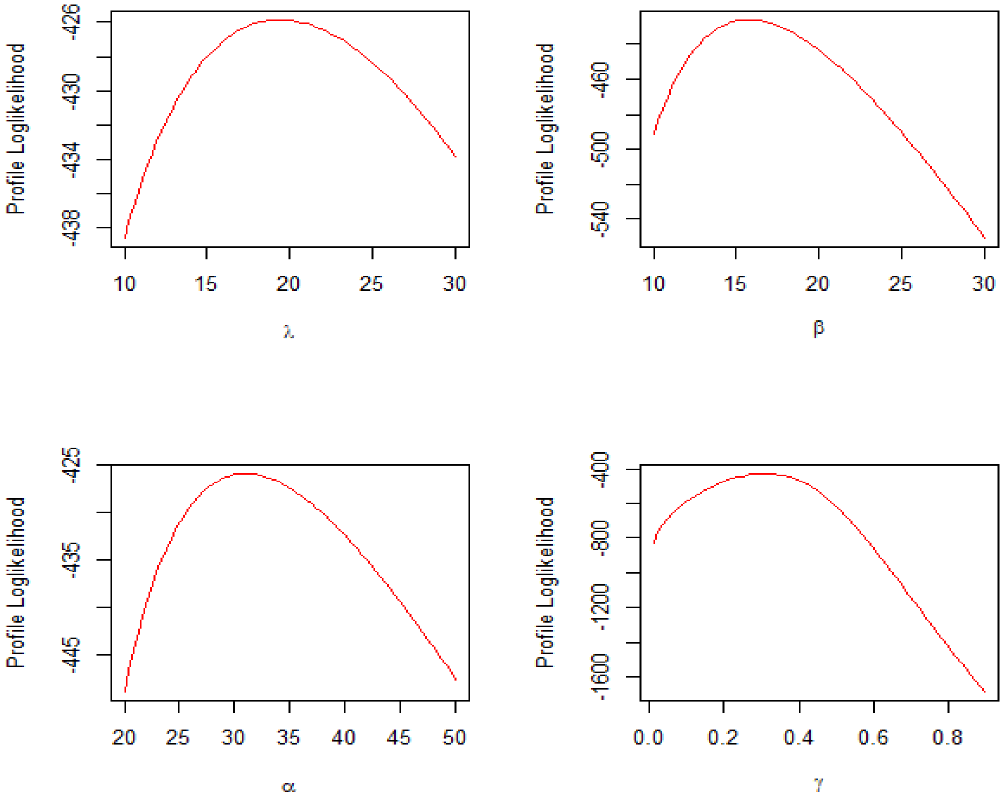

Figure 5.14: Profile log-likelihood plot of CW parameters for data2

5.9 Summary

The focus of most researchers is on developing new families that are generalizations of existing distributions to provide better fit for the modeling of life data. In this chapter, a new family of distributions called the CG family was developed and studied. Statistical properties, such as; quantile functions, moments, incomplete moments, generating function, entropies, stochastic ordering and order statistics, of the new family were derived. Estimators for the parameters of the new family were developed using the method of maximum likelihood. A demonstration of the application of the special distribution developed from the family was carried out using two real datasets. A comparison of the results with that of other existing distributions showed that the special distribution developed from the CG family provide a better parametric fit to these datasets.

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CHAPTER SIX

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.1 Introduction

The summary, conclusions and recommendations for future works are presented in this chapter.

6.2 Summary

Classical distributions do not always provide reasonable fit to all forms of datasets, hence, the quest to generate distributions with more desirable and flexible properties that can model real life datasets of varying shapes of density and failure rate functions. Currently, the focus is on the developing new families that are generalizations of existing distributions.

The study proposed two new generators of statistical distributions, OC and CG families of distributions, for generalizing existing distributions. The generalization approach used for obtaining the generators is the T-X approach, using the tranformations $W[G(x)] = G(x)$ and $W[G(x)] = \frac{G(x)}{1-G(x)}$ for the OC and CG families of distributions respectively.

The pdf, cdf and hazard functions of the generators were developed and the mixture representation of the density were derived. Other statistical properties, such as; the quantile functions, moments, generating functions, order statistics, inequality measures, entropies, mean residual life and stress strength reliability were also derived. The parameters of the generators were then estimated using maximum likelihood estimation method.

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New distributions were developed from the generators. These include; OCB, OCL, OCW, CB, CK and CW distributions. The pdf, cdf, hazard and quantile functions of the special distributions were established, whilst their skewness, kurtosis, density and hazard functions, were plotted, to give a graphical presentation of some of the type of datasets these distributions are capable of modeling. These plots revealed that the distributions are suitable for modeling lifetime datasets of varying shapes of density and hazard rate functions (both monotonic and non-monotonic shapes) exhibiting varying degrees of skewness and kurtosis. Simulations were then carried out to investigate the properties of the estimators of the parameters of some of the distributions developed, CW and OCL distributions.

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The usefulness of the new distributions developed was then demonstrated using four datasets. Two datasets, Data 1 and 2, were used to demonstrate the application of the distributions developed from the OC generator, thus OCB, OCL and OCW distributions. The other two datasets, Data 3 and 4, were then used to demonstrate the applications of the distributions developed from the CG family thus CB, CK and CW distributions in modeling real life datasets.

Exploratory analysis of Data 1 and 2 revealed that, Data 1 is left-skewed, heavy-tailed (leptokurtic) and exhibits a modified bathtub shaped failure rate. Data 2 is left-skewed, light-tailed (platykurtic) and exhibits a bathtub shaped failure rate. The maximum likelihood estimates of the parameters for the models were obtained by maximizing their log-likelihood functions. To examine how well the dataset corresponded to the fitted distributions, the AD, KS and CM tests were carried out, testing the hypothesis that; the data follow the specified distribution against the alternate that it does not. The results indicated that all the distribution provided good-fits to the datasets.

Comparative analysis were also carried out using the log-likelihood and the information criteria; AIC, BIC and CAIC. The performances of OCB, OCL and OCW distributions were compared with that of Chen and NGW distributions. From

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the results, OCL distribution outperformed the rest of the models as it had the highest log-likelihood and the lowest values for all the measures of the information criteria considered for both datasets. This was further graphically supported by the plots of the histograms and empirical cdf of the fitted distributions. The comparatively closer clustering of the observations of the OCL distribution along the diagonals of its P-P plots for both datasets also atest to the fact. A profile log-likelihood plot of OCL distribution's parameters showed that, its estimated parameter values are the maxima.

Data 3 and 4 were used to demonstrate the applications of CB and CW distributions. Preliminary analysis of the dataset revealed that both datasets were positively skewed, light-tailed (platykurtic) and had increasing hazard rates. Goodness-of-fit tests carried out showed that all the distributions fitted provided a good fit to the datasets. A comparison of the performances of CB and CW distributions against EC, EW and KEC distributions showed that, the CW distribution provided a comparatively better fit for the datasets, than the rest of the models. This was further supported by the plots of the histogram and empirical cdfs, and the P-P plots of fitted distributions for both datasets. Profile log-likelihood plot of CW distribution's parameters indicated that, its estimated parameter values are the maxima.

6.3 Conclusion

The study developed two new generators of statistical distributions, OC and CG families of distributions, for generalizing existing distributions to improve upon their flexibility in modeling real datasets.

The new families developed have closed forms of quantile making simulations possible and very easy. The hazard functions exhibit various shapes of failure rates, both monotonic and non-monotonic failure rates. This makes them suitable for modeling datasets of varying kinds.

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The OC and CG families of distributions were then used to modify the Burr III, Lomax, Kumaraswamy and Weibull distributions. Six special distributions were developed; OCB, OCL, OCW, CB, CK and CW distributions. The developed distributions are very flexible in modeling datasets, as they exhibit varying shapes of density and failure rates, for different combinations of parameter values. Also, the maximum likelihood estimators of these distributions are consistent as shown by the results of simulations carried out.

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Finally, the developed distributions are very useful in modeling real dataset and provide a consistently better parametric fit to some specific datasets than some existing candidate distributions.

6.4 Recommendations

This study considered only two generators, other generators of the Chen distribution can be developed by considering other methods of developing new distributions or using other transformations of the T-X approach.

A detailed study of the new distributions is needed to investigate their characterization and statistical properties including moments, generating functions, order statistics and stress strength reliability among others.

The study used only complete samples for the demonstration of the applications of the developed distributions, however some studies sometimes result in the generation of incomplete samples. Hence, further studies should consider the use of incomplete samples for the application.

Finally, the study only took into consideraton univariate situation, however a phenomenon may be influenced by more than one independent variable, hence a bivariate or multivariate extension of this work should be considered for further studies. Thus the developed distributions may be used to develop parametric regression models for analyzing bivariate or multivariate datasets.

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6.5 Major Contributions

The study proposed two new generators based on the Chen distribution for generalizing existing distributions to improve upon their flexibility in modeling datasets.

The statistical properties were derived and their parameters estimated. New distributions were proposed from the generators and a demonstration of their application shown using real life datasets.

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APPENDIX

A. Data and Source

1. Data 1 consists lifetimes of 50 components, given by Aarset (1987).

Table 6.1: Data1: Lifetimes of 50 components

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18
18	18	21	32	36	40	45	46	47	50	55	60	63	63	67	67
67	67	72	75	79	82	82	83	84	84	84	85	85	85	85	85
86	86														

2. Data 2 represents the lifetime of a certain device given by Sylwia (2007).

Table 6.2: Data 2: Lifetime of a certain device

0.0094	0.05	0.4064	4.6307	7.1645	7.2316	8.2616	9.2662
9.3812	9.5223	9.8783	10.4791	11.076	11.325	11.5284	11.9226
12.0294	12.5381	12.8049	13.4615	13.853	5.1741	5.8808	6.3348
10.4077	10.0192	9.9346	12.1835	12.074	12.3549		

3. Data 3 consists the fatigue times of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second found in Birnbaum and Saunders (1969).

Table 6.3: Data 3: Fatigue time of 101 6061-T6 aluminum coupons

70	90	96	97	99	100	103	104	104	105	107	108	108	108
109	109	112	112	113	114	114	114	116	119	120	120	120	121 121
121	123	124	124	124	124	124	128	128	129	129	130	130	130
131	131	131	131	131	132	132	132	133	134	134	134	134	134
136	136	137	138	138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151	151	152	155	156
157	157	157	157	158	159	162	163	163	164	166	166	168	170
174	196	212											

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4. Data 4 consists the survival times (in days) of 72 guinea pigs injected with different amount of virulent tubercle bacilli studied by Bjerkedal (1960).

Table 6.4: Data 4: Survival times of guinea pigs injected with different amount of tubercle bacilli.

10	33	44	56	59	72	74	77	92	93	96	100	100	102
105	107	107	108	108	108	109	112	113	115	116	120	121	122
122	124	130	134	136	139	144	146	153	159	160	163	163	168
171	172	176	183	195	196	197	202	213	215	216	222	230	231
240	245	251	253	254	255	278	293	327	342	347	361	402	432
458	555												

B. Publications

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Annals of Data Science
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Odd Chen-G Family of Distributions

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Abstract
Classical distributions do not always provide reasonable fit to all forms of datasets, hence the need to generalize existing distributions to enhance their flexibility in modeling of data. The study developed the odd Chen-G family of distributions. It derives the statistical properties of the new family such as the quantile, moments, and order statistics. Though capable of generalizing other distributions, the study proposed three special distributions; odd Chen Burr III, odd Chen Lomax and odd Chen Weibull distributions. The application of the new family is then demonstrated using real data.

Keywords Odd · Chen · Lomax · Statistical distribution · Quantile

Mathematics Subject Classification 62E15 · 60E05

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1 Introduction

There are numerous univariate statistical distributions in literature for modeling dataset, notably among the classical continuous parametric ones are Weibull, gamma, beta, log-normal and exponential distributions. However, the complex nature of certain researches often results in datasets which is difficult to model using these classical distributions as they do not always produce reasonable fit. To achieve flexibility in the modeling of datasets, researchers are continuously developing new distributions, which are generalizations of existing ones using techniques such as exponentiation and the T-X approach. Some generalized families of distributions in literature

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include; Zubair-G [1], extended odd Fréchet-G [2], Kumaraswamy-G [3], beta-G [4], transformed-transformer (T-X) family [5], exponentiated T-X [6], exponentiated generalized T-X [7], Weibull-G [8], odd generalized exponential family [9], odd Topp-Leone odd log-logistic-G family [10] and odd Fréchet-G family [11].

Though a two-parameter distribution, the Chen distribution has the ability to model bathtub shaped failure rate functions among others desirable properties [12]. It however lacks a scale parameter which makes it less flexible for modeling datasets [13]. The cumulative distribution function (cdf), $G(x)$ for Chen distribution is given by

$$G(x) = 1 - e^{\lambda\left(1-e^{x^{\beta}}\right)}, x > 0, \lambda > 0, \beta > 0. \tag{1}$$

This study proposes a new family of distribution called the odd Chen-G (OCG) family of distributions using the T-X approach.

Let $G(x;\psi)$ be baseline cdf and ψ be a vector of associated parameters, the cdf $F(x)$ of the OCG family of distributions is defined as

$$F(x) = \int_{0}^{\frac{G(x;\psi)}{1-G(x;\psi)}} f(t)dt = 1 - e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}, x > 0,\ \lambda > 0,\ \beta > 0, \tag{2}$$

where λ and β are extra shape parameters. Differentiating the cdf in Eq. (2), the probability density function (pdf) $f(x)$ of the family is obtained as

$$f(x) = \lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1 - G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}, x > 0. \tag{3}$$

The corresponding survival $S(x)$ and hazard $h(x)$ functions are respectively given by

$$S(x) = e^{\lambda\left(1-e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)}, x > 0 \tag{4}$$

and

$$h(x) = \lambda\beta g(x;\psi)G(x;\psi)^{\beta-1}[1 - G(x;\psi)]^{-(\beta+1)}e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}, x > 0\, \lambda > 0, \beta > 0. \tag{5}$$

For simplicity, let $G(x;\psi)$ be denoted as $G(x)$ in the rest of paper. Mixture representation of the pdf and derivation of statistical properties of the OCG family of distributions is presented in Sect. 2, followed by the estimation of its parameters in Sect. 3. In Sect. 4, some proposed special distributions from the OCG family are presented. The properties of estimators of the parameters of these special distributions are examined using simulations in Sect. 5 and demonstrations of the usefulness of the

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special distributions using real life datasets are shown in Sect. 6. Concluding remarks are captured in Sect. 7.

2 Mixture Representation and Statistical Properties

Expansions of the density function of the family are made in this section. Also, statistical properties of the family such as quantile, moments, order statistics and entropies are derived.

2.1 Mixture Representation

The mixture representation of the pdf is essential in the derivation of the statistical properties of the OCG family of distributions. After applying Taylor series expansion, the OCG pdf in Eq. (3) becomes

$$f(x) = \lambda\beta g(x)e^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i\lambda^i}{i!}\frac{(i+1)^j}{j!}G(x)^{\beta(j+1)-1}[1-G(x)]^{-[\beta(j+1)+1]}. \quad (6)$$

Further expanding Eq. (3) using the generalized binomial series expansion,

$$(1-z)^{-a} = \sum_{k=0}^{\infty}\binom{a+k-1}{k}z^k,\ |z| \leq 1, a < 0,$$

the expression for $f(x)$ becomes

$$f(x) = \lambda\beta e^{\lambda}g(x)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i\lambda^i}{i!}\frac{(i+1)^j}{j!}\binom{\beta(j+1)+k}{k}G(x)^{\beta(j+1)+k-1}.$$

Assuming a an integer in the binomial series expansion, the expression of the mixture representation of the pdf for the OCG family is

$$f(x) = \lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\nu_{ijkmq}\,g(x)G(x)^q, \quad (7)$$

where

$$\nu_{ijkmq} = \frac{(-1)^{i+m}\lambda^i(i+1)^j}{i!\,j!}e^{\lambda}\binom{\beta(j+1)+k}{k}\binom{\beta(j+1)+k-1}{m}\binom{m}{q}.$$

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Equation (7) expresses the pdf of the OCG family as a product of its parameters and sum of the product of the pdf and weighted power series of the baseline distribution function. Also, expressing $f(x)$ in terms of exponentiated-G (expo-G) density yields

$$f(x) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu^{*}_{ijkmq}\pi_{q+1}(x), \tag{8}$$

where

$$\nu^{*}_{ijkmq} = \frac{\nu_{ijkmq}}{q+1} \text{ and } \pi_{q+1}(x) = (q+1)g(x)(G(x))^{q}$$

is the expo-G density function with power parameter $(q + 1)$.

2.2 Quantile Function

Random number generation for simulation purposes is one of the essential uses of the quantile function.

Proposition 1 *The quantile function for the OCG family of distributions is given by*

$$Q_G(u) = G^{-1}\left[\frac{\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}{1+\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}\right], \quad 0 < u < 1. \tag{9}$$

Proof The quantile function $Q_G(u)$ of a random variable X is defined as the inverse of the cdf. Hence, replacing x with x_u, $u \in (0, 1)$ in Eq. (2), equating $F(x_u)$ to u and making x_u the subject yields the quantile function.

The median of the family is obtained when $u = 0.5$. □

2.3 Moments, Moment Generating Functions and Incomplete Moments

Moments are useful in the study of the characteristics of a distribution such as skewness and kurtosis whilst incomplete moments are key in computing measures such as Lorenz and Bonferroni curves.

2.3.1 Moments

Proposition 2 *The* rth *non-central moment for the OCG family of distributions is*

$$\mu_r^{'} = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\tau_{(r,q)}, \tag{10}$$

where

$$\tau_{(r,q)} = \int_{-\infty}^{\infty} x^r g(x)(G(x))^q dx$$

is the weighted moment of the baseline distribution $G(x)$.

Proof The rth non-central moment for a random variable X is defined as $\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx$. Substituting the mixture density $f(x)$ from Eq. (7) into the definition yields

$$\mu_r^{'} = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{\infty} x^r g(x)(G(x))^q dx.$$

Alternatively, the rth non-central moment is defined in terms of the quantile function as

$$\mu_r^{'} = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{0}^{1} u^q Q_G^r(u)du, \; 0 < u < 1. \tag{11}$$

where $u = G(x)$ and $Q_G(u)$ is the quantile function of the baseline distribution. □

2.3.2 Moment Generating Functions

Proposition 3 *The moment generating function* $\mathrm{M}_X(t)$ *for the OCG family of distributions is given by*

$$\mathrm{M}_X(t) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{r=0}^{\infty} \frac{t^r}{r!} \nu_{ijkmq} \tau_{(r,q)}. \tag{12}$$

Proof Generally, the moment generating function for a random variable X is defined as $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$. Hence, expanding $\mathrm{M}_X(t)$ using Taylor series yields

$$\mathrm{M}_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x)dx.$$

Subsequently, substituting the expression for the rth non-central moment into the definition of $\mathrm{M}_X(t)$ yields

$$\mathrm{M}_X(t) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{r=0}^{\infty} \frac{t^r}{r!} \nu_{ijkmq} \tau_{(r,q)}.$$

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$M_X(t)$ can further be expressed in terms of quantile function as

$$M_X(t) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{r=0}^{\infty} \nu_{ijkmq} \int_0^1 e^{tQ_G(x)} u^q du, 0 < u < 1. \quad (13)$$

□

2.3.3 Incomplete Moments

Incomplete moments play a key role in the computation of statistical measures such as the mean deviations about the mean and median.

Proposition 4 *The incomplete moments $M_r(y)$ of the OCG family of distributions is given by*

$$M_r(y) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{y} x^r g(x) G^q(x) dx \quad (14)$$

Proof The incomplete moments of a random variable X is defined as $M_r(y) = \int_{-\infty}^{y} x^r f(x)dx$. Substituting $f(x)$ in Eq. (7) into the expression yields

$$M_r(y) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{-\infty}^{y} x^r g(x) G^q(x) dx.$$

Alternatively, $M_r(y)$ may be expressed in terms of quantile function as

$$M_r(y) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq} \int_{0}^{G(y)} u^q Q_G^r(u) du. \quad (15)$$

□

2.4 Order Statistics

Order statistics are very useful in many areas of statistical theory most especially extreme-value theory. The pdf for the pth order statistic $X_{p:n}$, of the ordered random sample $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ of size n is denoted by $f_{X_{p:n}}(x)$.

Proposition 5 *The pdf for the* pth *order statistic of the OCG family of distributions is the obtained as*

$$f_{X_{p:n}}(x) = \lambda\beta \sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=o}^{m} D_{ijklmn} g(x) G(x)^n, \quad (16)$$

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where

$$D_{ijklmn} = \frac{(-1)^{i+j+m} n!\,(\lambda(n-p+i+1))^j (j+1)^k e^{\lambda(n-p+i+1)}}{j!\,k!\,(p-1)!\,(n-p)!} \times \binom{p-1}{i}\binom{\beta(k+1)+l}{l}\binom{\beta(k+1)+l-1}{m}\binom{m}{n}.$$

Proof The pdf for the p^{th} order statistic $X_{p:n}$, of a random sample $X_1, X_2, \ldots, X_n$ of size $n, f_{X_{p:n}}(x)$, is generally defined as

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}[F(x)]^{p-1}[1-F(x)]^{n-p} f(x),\; p = 1, 2, \ldots, n.$$

Expanding $[F(x)]^{p-1}$ in the definition of $f_{X_{p:n}}(x)$ using binomial series expansion yields,

$$[F(x)]^{p-1} = \sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}[1-F(x)]^i.$$

Substituting it back into the expression of $f_{X_{p:n}}(x)$ yields

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}[S(x)]^{n-p+i} f(x),$$

where

$$[S(x)]^{n-p+i} = [1-F(x)]^{n-p+i} = e^{\lambda(n-p+i)\left(1-e^{G(x)^\beta}\right)}.$$

Algebraically manipulating

$$[S(x)]^{n-p+i} f(x) = \lambda\beta g(x)G(x)^{\beta-1}e^{G(x)^\beta}e^{\lambda(n-p+i+1)\left(1-e^{G(x)^\beta}\right)}$$

using Taylor series expansion yields;

$$[S(x)]^{n-p+i} f(x) = \lambda\beta g(x)G(x)^{\beta-1}e^{\lambda(n-p+i+1)} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j[\lambda(n-p+i+1)]^j}{i!}\frac{(j+1)^k}{k!}G(x)^{\beta k}.$$

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Further applying binomial series expansion gives;

$$[S(x)]^{n-p+i} f(x) = \lambda\beta g(x) e^{\lambda(n-p+i+1)} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(-1)^{j+m}[\lambda(n-p+i+1)]^j}{i!}\frac{(j+1)^k}{k!} \times \binom{\beta(k+1)+l}{l}\binom{\beta(k+1)+l-1}{m}\binom{m}{n} G(x)^n.$$

Subsequently, substituting the expression of $[S(x)]^{n-p+i} f(x)$ into that of $f_{X_{p:n}}(x)$ yields

$$f_{X_{p:n}}(x) = \lambda\beta \sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=o}^{m} D_{ijklmn} g(x) G(x)^n.$$

□

2.4.1 Moments of Order Statistics

Proposition 6 *The* rth *non-central moment of the* pth *order statistic,* $E(X^r_{p:n})$*, of the OCG family of distributions is given by,*

$$E(X^r_{p:n}) = \lambda\beta \sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m} D_{ijklmn} \tau_{r,n}, \tag{17}$$

where $\tau_{r,n} = \int_{-\infty}^{\infty} x^r g(x) G(x)^n dx$ *is the probability weighted moment of the baseline distribution.*

Proof The rth moment of the pth order statistic of a random variable is defined as $E(X^r_{p:n}) = \int_{-\infty}^{\infty} x^r f_{X_{p:n}}(x)dx$. Hence, substituting the pdf for the pth order statistic in Eq. (16) into the expression of $E(X^r_{p:n})$, completes the proof. □

2.5 Stochastic Ordering

Stochastic ordering is used to show the ordering mechanism of a dataset. A random variable X with cdf $F_X(x)$ is less than Y with cdf $F_Y(x)$ in likelihood ratio order ($X \leq_{lr} Y$), if the function $f_X(x)/f_Y(x)$ is decreasing for all x.

Proposition 6 *Let* $X \sim OCG(x; \lambda_1, \beta, \psi)$ *and* $Y \sim OCG(x; \lambda_2, \beta, \psi)$*. Then* X *is less than* Y *in likelihood ratio order* ($X \leq_{lr} Y$) *if* $\lambda_2 < \lambda_1$.

Proof To determine whether the ratio of the pdfs of X and Y,

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_1-\lambda_2)\left(1-e^{\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}\right)}$$

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is an increasing or decreasing function, we take the differential of its logarithm. If $\lambda_2 < \lambda_1$ then $\frac{d}{dx}\left(\log\left(\frac{f_X(x)}{f_Y(x)}\right)\right) < 0$ for all x. Hence, X is smaller than Y in likelihood ratio order ($X \leq_{lr} Y$) and by implication X is smaller than Y in stochastic order ($X \leq_{st} Y$). □

2.6 Inequality Measure

Several fields like insurance, econometrics and reliability employ Lorenz and Bonferroni curves in the study of inequality measures like income and poverty.

2.6.1 Lorenz Curve

Lorenz curve is defined as $\mathrm{L}_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx$, hence for the OCG family of distributions, it is given by

$$\mathrm{L}_F(y) = \frac{\lambda\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\int_{-\infty}^{y} xg(x)G^q(x)dx. \quad (18)$$

Alternatively, it can be expressed in terms of quantile function as

$$L_F(y) = \frac{\lambda\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\int_{0}^{G(y)} u^q Q_G(u)du. \quad (19)$$

2.6.2 Bonferroni Curve

The Bonferroni curve is another inequality measure given by $B_F(y) = \frac{\mathrm{L}_F(y)}{F(y)}$. From Eq. (19), the Bonferroni curve for the OCG family of distributions is obtained as

$$B_F(y) = \frac{\lambda\beta}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\int_{-\infty}^{y} xg(x)G^q(x)dx. \quad (20)$$

2.7 Mean Residual Life

The mean residual life is the expected residual life or the average survival time of the component after it exceeds a specific time y. It plays a very useful role in reliability studies.

Proposition 7 *The mean residual life of a CG random variable Y is given by*

$$\overline{M}(y)=\frac{1}{F(y)}\left[\mu - \lambda\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\int_{-\infty}^{y} xg(x)G^q(x)dx\right] - y. \quad (21)$$

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Proof The mean residual life is defined as $\overline{M}(y) = E(X - y/X > y)$, thus

$$\overline{M}(y) = \frac{1}{F(y)}\left[\mu - \int_{-\infty}^{y} xf(x)dx\right] - y.$$

Substituting the mixture density $f(x)$ in Eq. (7) into the expression of $\overline{M}(y)$ yields the proof. □

2.8 Entropy

Entropy measures the variation or uncertainty of a random variable. Its application spans across several disciplines some of which include; econometrics, engineering, probability theory and science in general.

2.8.1 Rényi's Entropy

The application of entropy as a measure of variation or uncertainty of a random variable can be seen in many discipline some of which include; engineering, econometrics and financial mathematics.

Proposition 8 *Renyi's entropy for the CG random variable is given by*

$$\mathrm{I}_R(\delta) = \frac{1}{1-\delta}\log\left[(\lambda\beta)^{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\eta_{ijklm}\int_{-\infty}^{\infty} g(x)^{\delta}G(x)^{m}dx\right],\ \delta \neq 1, \delta > 0 \tag{22}$$

where

$$\eta_{ijklm} = \frac{(-1)^{i+l}(\lambda\delta)^{i}(i+\delta)^{j}e^{\lambda\delta}}{i!\,j!}\times \binom{\beta(j+\delta)+\delta+k-1}{k}\binom{\beta(j+\delta)-\delta+k}{l}\binom{l}{m}.$$

Proof Let X be a random variable with pdf $f(x)$, the Rényi's entropy [14] is given by

$$\mathrm{I}_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty} f^{\delta}(x)dx\right],\ \delta \neq 1, \delta > 0.$$

From Eq. (3),

$$f^{\delta}(x) = \left[\lambda\beta g(x)G(x)^{\beta-1}[1-G(x)]^{-(\beta+1)}e^{\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}e^{\lambda\left(1-e^{\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}\right)}\right]^{\delta}$$

Expanding $f^{\delta}(x)$ using a similar concept as used in obtaining the mixture representation of the density function yields

$$f^{\delta}(x) = (\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \eta_{ijklm}\, g(x)^{\delta} G(x)^{m},$$

where

$$\eta_{ijklm} = \frac{(-1)^{i+l}(\lambda\delta)^{i}(i+\delta)^{j} e^{\lambda\delta}}{i!\,j!} \times \binom{\beta(j+\delta)+\delta+k-1}{k}\binom{\beta(j+\delta)-\delta+k}{l}\binom{l}{m}.$$

Substituting $f^{\delta}(x)$ into the expression for $\mathrm{I}_R(\delta)$ yields

$$\mathrm{I}_R(\delta) = \frac{1}{1-\delta}\log\left[(\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \eta_{ijklm} \int_{-\infty}^{\infty} g(x)^{\delta} G(x)^{m} dx\right].$$

□

2.8.2 Shannon's Entropy

The Shannon's entropy [14] for a random variable X with pdf $f(x)$ is a special case of the Rényi's entropy when $\delta \uparrow 1$. It is defined as $\eta_X = E(-\log f(x))$. For the OCG family of distribution it is given by

$$\eta_X = E\left[-\log\left(\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \nu_{ijkmq}\, g(x) G(x)^{q}\right)\right]. \tag{23}$$

2.8.3 Delta Entropy

The δ-entropy is given by

$$H(\delta) = \frac{1}{1-\delta}\log\left[1 - \int_{-\infty}^{\infty} f^{\delta}(x) dx\right].$$

Hence the δ-entropy for the OCG family of distributions is given by

$$\mathrm{I}_R(\delta) = \frac{1}{1-\delta}\log\left[1 - (\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \eta_{ijklm} \int_{-\infty}^{\infty} g(x)^{\delta} G(x)^{m} dx\right],\ \delta \neq 1, \delta > 0. \tag{24}$$

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2.9 Stress–Strength Reliability

The stress strength reliability is the probability of the component to perform without fail, a specified function under specified conditions for a given level of stress.

Proposition 9 *The Stress strength reliability R of the OCG family is given by*

$$R = 1 - \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \varphi_{ijklm} \int_{-\infty}^{\infty} g(x)G(x)^m dx, \tag{25}$$

where

$$\varphi_{ijklm} = \frac{(-1)^{i+l}(2\lambda)^i(i+1)^j e^{2\lambda}}{i!\,j!} \binom{\beta(j+1)+k}{k}\binom{\beta(j+1)+k-1}{l}\binom{l}{m}.$$

Proof Suppose $X_1 \sim (\lambda, \beta, \psi)$ is a strength random variable and $X_2 \sim (\lambda, \beta, \psi)$ is a stress random variable both from the OCG family. The stress strength reliability is defined by

$$R = P(X_2 < X_1) = \int_{-\infty}^{\infty} f(x)F(x)dx = 1 - \int_{-\infty}^{\infty} f(x)S(x)dx.$$

Algebraically manipulating $f(x)S(x)$ in the expression of R in a similar manner as the mixture of $f(x)$ in Eq. (7) yields

$$f(x)S(x) = \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \varphi_{ijklm} g(x)G(x)^m,$$

where

$$\varphi_{ijklm} = \frac{(-1)^{i+l}(2\lambda)^i(i+1)^j e^{2\lambda}}{i!\,j!} \binom{\beta(j+1)+k}{k}\binom{\beta(j+1)+k-1}{l}\binom{l}{m}.$$

Substituting the expression of $f(x)S(x)$ obtained back into R yields the proof. □

3 Parameter Estimation

The parameters of the OCG family are estimated in this section using the method of maximum likelihood. Given a random sample $x_1, x_2, \ldots, x_n$ of size n with parameters

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λ, β and ψ from the OCG family of distribution. Let $\upsilon = (\lambda, \beta, \psi)^T$ be a $(p \times 1)$ parameter vector, the total log-likelihood function is given by

$$\ell = n \log \lambda\beta + \sum_{i=1}^{n} \log g(x; \psi) + (\beta - 1) \sum_{i=1}^{n} \log G(x; \psi) + \lambda \sum_{i=1}^{n} \left(1 - e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right)$$
$$- (\beta + 1) \sum_{i=1}^{n} \log[1 - G(x; \psi)] \sum_{i=1}^{n} \left(\frac{G(x; \psi)}{1 - G(x; \psi)}\right)^{\beta}. \quad (26)$$

partially differentiating the likelihood function yields the components of the score function $U(\upsilon) = (\partial\ell/\partial\lambda, \partial\ell/\partial\beta, \partial\ell/\partial\psi)^T$ as follows

$$\frac{d\ell}{d\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \left(1 - e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}\right), \quad (27)$$

$$\frac{d\ell}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log G(x; \psi) - \sum_{i=1}^{n} \log[1 - G(x; \psi)]$$
$$+ \sum_{i=1}^{n} \left(\frac{G(x; \psi)}{1 - G(x; \psi)}\right)^{\beta} \log\left(\frac{G(x; \psi)}{1 - G(x; \psi)}\right)$$
$$- \lambda \sum_{i=1}^{n} \left(\frac{G(x; \psi)}{1 - G(x; \psi)}\right)^{\beta} \log\left(\frac{G(x; \psi)}{1 - G(x; \psi)}\right) e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}, \quad (28)$$

$$\frac{d\ell}{d\psi} = \sum_{i=1}^{n} \frac{g_k^{'}(x; \psi)}{g(x; \psi)} + (\beta - 1) \sum_{I=1}^{n} \frac{G_k^{'}(x; \psi)}{G(x; \psi)} + (\beta + 1) \sum_{i=1}^{n} \frac{G_k^{'}(x; \psi)}{[1 - G(x; \psi)]}$$
$$+ \beta \sum_{i=1}^{n} \frac{G_K^{'}(x; \psi) G(x; \psi)^{\beta-1}}{[1 - G(x; \psi)]^{\beta+1}} - \lambda\beta \sum_{i=1}^{n} \frac{G_K^{'}(x; \psi) G(x; \psi)^{\beta-1}}{[1 - G(x; \psi)]^{\beta+1}} e^{\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{\beta}}, \quad (29)$$

where $g_K^{'}(x; \psi) = \frac{dg(x;\psi)}{d\psi}$, $g_K^{''}(x; \psi) = \frac{d^2 g(x;\psi)}{d\psi^2}$, $G_K^{'}(x; \psi) = \frac{dG(x;\psi)}{d\psi}$ and $G_K^{''}(x; \psi) = \frac{d^2 G(x;\psi)}{d\psi^2}$.

The estimators of the parameters are then obtained by setting Eqs. (27), (28) and (29) to zero and solving them numerically using the iterative methods such as the Newton–Raphson type algorithms. The observed information matrix $J(\upsilon)$, is required for interval estimation of the parameters. It can be estimated as $J(\upsilon) = \frac{\partial^2 \ell}{\partial i \partial j}$ for $(i, j = \lambda, \beta, \psi)$ whose elements are evaluated numerically.

4 Some Special Distributions

Generalization of several distributions can be made using the OCG family of distributions. Three special distributions; odd Chen Burr III (OCB), odd Chen Lomax (OCL) and odd Chen Weibull (OCW) were developed in this section.

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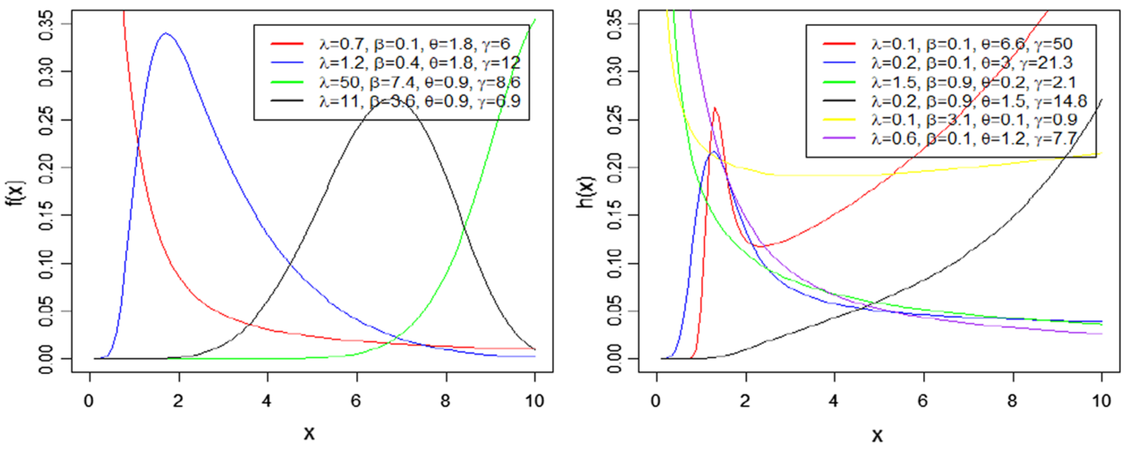

Fig. 1 plots of density and hazard rate functions of OCB distribution

4.1 Odd Chen Burr III Distribution

The cdf and pdf of Burr III (the baseline) distribution are respectively $G(x) = \left(1 + x^{-\theta}\right)^{-\gamma}$ and $g(x) = \gamma\theta x^{-\theta-1}\left(1 + x^{-\theta}\right)^{-\gamma-1}$, $x > 0, \theta > 0, \gamma > 0$. Substituting $G(x)$ and $g(x)$ into Eqs. (2), (3) and (4), respectively yields the cumulative distribution, probability density and hazard functions of the OCB distribution. The cdf and pdf of the OCB distribution are given by

$$F(x) = 1 - e^{\lambda\left(1 - e^{\left[\left(1+x^{-\theta}\right)^{\gamma}-1\right]^{-\beta}}\right)}, x > 0, \theta > 0, \beta > 0, \gamma > 0, \lambda > 0 \tag{30}$$

and

$$f(x) = \lambda\beta\gamma\theta x^{-(\theta+1)}\left(1 + x^{-\theta}\right)^{-(\gamma\ \beta+1)}\left[1 - \left(1 + x^{-\theta}\right)^{-\gamma}\right]^{-(\beta+1)} e^{\left[\left(1+x^{-\theta}\right)^{\gamma}-1\right]^{-\beta}} e^{\lambda\left(1-e^{\left[\left(1+x^{-\theta}\right)^{\gamma}-1\right]^{-\beta}}\right)}, x > 0. \tag{31}$$

Its hazard function is given by

$$h(x) = \lambda\beta\gamma\theta x^{-(\theta+1)}\left(1 + x^{-\theta}\right)^{-(\gamma\ \beta+1)}\left[1 - \left(1 + x^{-\theta}\right)^{-\gamma}\right]^{-(\beta+1)} e^{\left[\left(1+x^{-\theta}\right)^{\gamma}-1\right]^{-\beta}}, x > 0. \tag{32}$$

The OCB distribution exhibits increasing, decreasing, unimodal left and right skewed shapes of density function and for some selected values it exhibits bathtub, upside down bathtub, modified upside down bathtub, decreasing and increasing failure rates as shown by its density and hazard rate plots in Fig. 1.

The quantile function $Q_G(u)$ for the Odd Chen Burr III distribution is given by

$$Q_G(u) = \left\{\left[\frac{\left(\log\left(1 - \left(\frac{\log(1-u)}{\lambda}\right)\right)\right)^{\frac{1}{\beta}}}{1 + \left(\log\left(1 - \left(\frac{\log(1-u)}{\lambda}\right)\right)\right)^{\frac{1}{\beta}}}\right]^{-\frac{1}{\gamma}} - 1\right\}^{-\frac{1}{\theta}}, 0 < u < 1. \tag{33}$$

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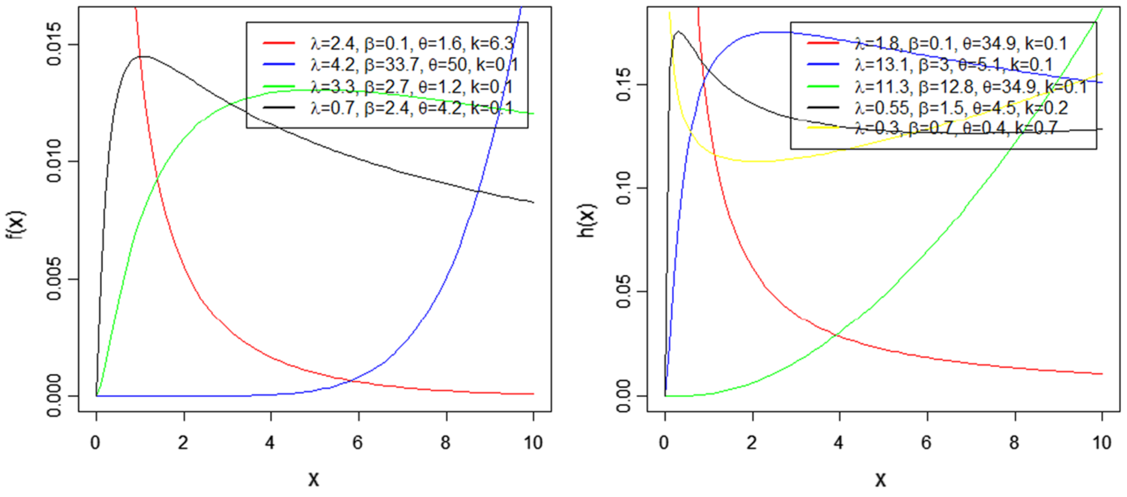

Fig. 2 Plots of density and hazard rate functions of OCL distribution

4.2 Odd Chen Lomax Distribution

The cdf and pdf of the Lomax distribution [15] is respectively given by $G(x) = 1 - (1+\theta x)^{-k}$ and $g(x) = \theta k(1+\theta x)^{-(k+1)}$, $x > 0, k > 0, \theta > 0$. The cdf and pdf of the OCL distribution is given by

$$F(x) = 1 - e^{\lambda\left(1-e^{\left[(1+\theta x)^{k}-1\right]^{\beta}}\right)}, x > 0, \lambda > 0, \theta > 0, \beta > 0, k > 0 \quad (34)$$

and

$$f(x) = \lambda\beta\theta k(1+\theta x)^{\beta k-1}\left[1-(1+\theta x)^{-k}\right]^{\beta-1} e^{\left[(1+\theta x)^{k}-1\right]^{\beta}} e^{\lambda\left(1-e^{\left[(1+\theta x)^{k}-1\right]^{\beta}}\right)}, x > 0. \quad (35)$$

Its hazard function is given by

$$h(x) = \lambda\beta\theta k(1+\theta x)^{\beta k-1}\left[1-(1+\theta x)^{-k}\right]^{\beta-1} e^{\left[(1+\theta x)^{k}-1\right]^{\beta}}, x > 0. \quad (36)$$

The density plot of the OCL distribution exhibit varying shapes such as increasing, decreasing and non monotonically increasing shapes among others as shown in Fig. 2. The hazard rate function exhibited upside down bathtub, decreasing and increasing failure rates for some selected values.

The quantile function for the Odd Chen Lomax distribution is obtained as

$$Q_G(u) = \frac{1}{\theta}\left[\left(\left(1-\left(\frac{\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}{1+\left(\log\left(1-\frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}\right)\right)\right)^{-\frac{1}{k}} - 1\right], 0 < u < 1. \quad (37)$$

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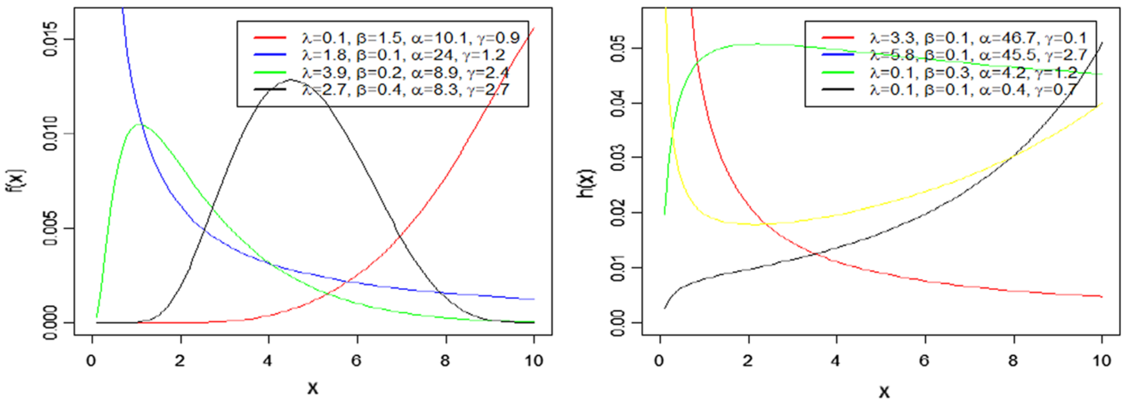

Fig. 3 Plots of the density and hazard rate function of OCW distribution

4.3 Odd Chen Weibull Distribution

The cdf and pdf of Weibull distribution are $G(x) = 1 - e^{-(\frac{x}{\alpha})^{\gamma}}$ and $g(x) = \left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1} e^{-(\frac{x}{\alpha})^{\gamma}}$ respectively. The cdf and pdf of OCW distribution is obtained as

$$F(x) = 1 - e^{\lambda\left[1-e^{\left(e^{(\frac{x}{\alpha})^{\gamma}}-1\right)^{\beta}}\right]}, x > 0, \alpha > 0, \beta > 0, \gamma > 0 \tag{38}$$

and

$$f(x) = \lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1 - e^{-(\frac{x}{\alpha})^{\gamma}}\right)^{\beta-1} e^{-(\frac{x}{\alpha})^{-\gamma\beta}} e^{\left(e^{(\frac{x}{\alpha})^{\gamma}}-1\right)^{\beta}} e^{\lambda\left(1-e^{\left(e^{(\frac{x}{\alpha})^{\gamma}}-1\right)^{\beta}}\right)}, x > 0. \tag{39}$$

Its hazard function is given by

$$h(x) = \lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1 - e^{-(\frac{x}{\alpha})^{\gamma}}\right)^{\beta-1} e^{-(\frac{x}{\alpha})^{-\gamma\beta}} e^{\left(e^{(\frac{x}{\alpha})^{\gamma}}-1\right)^{\beta}} x > 0. \tag{40}$$

A display of plots of the density and hazard rate functions of the OCW distribution are found in Fig. 3. The density plot shows shapes such as symmetric, unimodal right skewed, J and reversed J shapes. The hazard rate plot for some selected values exhibits increasing and decreasing failure rates, bathtub and upside down bathtub shapes.

The quantile function $Q_G(u)$ the Odd Chen Weibull distribution is given by

$$Q_G(u) = \alpha\left[-\log\left(1 - \frac{\left(\log\left(1 - \frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}{1 + \left(\log\left(1 - \frac{\log(1-u)}{\lambda}\right)\right)^{\frac{1}{\beta}}}\right)\right]^{\frac{1}{\gamma}}, 0 < u < 1. \tag{41}$$

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Table 1 Simulation results of AB, RMSE and CP for OCW distribution

Parameter	n	$(\theta = 0.3, \beta = 0.8, \gamma = 0.1, \lambda = 0.4)$			$(\theta = 0.9, \beta = 3.5, \gamma = 2.5, \lambda = 0.6)$		
		AB	RMSE	CP	AB	RMSE	CP
θ	50	0.5467	0.7802	0.8880	2.8948	3.8475	0.9840
	150	0.3836	0.6492	0.8673	1.6762	2.4370	0.9560
	300	0.2496	0.5282	0.8633	1.1345	1.6802	0.9467
	600	0.1365	0.3922	0.8600	0.7915	1.1433	0.9600
	1000	0.0872	0.3048	0.8853	0.6119	0.8900	0.9540
β	50	0.0388	0.1555	0.9833	− 2.1826	2.5354	0.4593
	150	0.0249	0.0812	0.9740	− 1.8235	2.2538	0.5820
	300	0.0165	0.0567	0.9820	− 1.6394	2.0520	0.6387
	600	0.0117	0.0371	0.9813	− 1.4499	1.8822	0.6523
	1000	0.0072	0.0263	0.9727	− 1.2787	1.7325	0.6440
γ	50	− 0.0129	0.0962	0.9999	246.9147	966.0244	0.9767
	150	− 0.0081	0.0732	0.9967	28.4697	107.9522	0.9887
	300	− 0.0014	0.0620	0.9740	7.6163	21.8327	0.9953
	600	0.0021	0.0479	0.9620	2.9083	4.8612	0.9999
	1000	0.0016	0.3880	0.9553	2.0458	3.2785	0.9900
λ	50	0.1021	0.4368	0.8947	0.8223	17.0391	0.6080
	150	0.0150	0.2060	0.9267	− 0.1217	0.7355	0.6553
	300	− 0.0064	0.1398	0.9307	− 0.1766	0.4760	0.7100
	600	− 0.0104	0.0972	0.9433	− 0.1953	0.3604	0.7367
	1000	− 0.0084	0.0772	0.9393	− 0.1898	0.3135	0.7487

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5 Simulation

Validation of the maximum likelihood estimators is carried out in this section using Monte Carlo simulations. This is done using the estimators of the OCW distribution. Random numbers from the OCW distribution are generated using the OCW quantile function in Eq. (43). Setting the initial parameter values; $\theta = 0.3$, $\beta = 0.8$, $\gamma = 0.1$ and $\lambda = 0.4$, for sample sizes $n = 50, 150, 300, 600, 1000$, the simulations are repeated 1500 times for each sample. Repeating similar sample sizes for the initial parameter values; $\theta = 0.9, \beta = 3.5, \gamma = 2.5$ and $\lambda = 0.6$, the simulations are repeated for each sample another 1500 times. The root mean square error (RMSE), the average bias (AB) and coverage probability (CP) for the estimators of the parameters at 95% confidence intervals are presented in Table 1. From Table 1 it is observed that there is convergence of the RMSE and AB in all cases. Thus they decrease to zero(0) as the sample size increases. The CPs are also observed to be close the nominal value of 0.95. This emphasizes the effectiveness of the method of maximum likelihood in estimating the parameters of the OCW distribution.

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Table 2 Lifetimes of 50 components (Data 1)

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12
18	18	18	18	18	21	32	36	40	45	46	47	50
55	60	63	63	67	67	67	67	72	75	79	82	82
83	84	84	84	85	85	85	85	85	86	86		

Table 3 Lifetime of a certain device (Data 2)

0.0094	0.0500	0.4064	4.6307	7.1645	7.2316	8.2616	9.2662	9.3812	9.5223
9.8783	10.4791	11.0760	11.3250	11.5284	11.9226	12.0294	12.5381	12.8049	13.4615
13.8530	5.1741	5.8808	6.3348	10.4077	10.0192	9.9346	12.1835	12.0740	12.3549

6 Applications

In this section, real life datasets are used to demonstrate the applications of the OCB, OCL and OCW distributions in providing good parametric fit. The maximum likelihood estimates for the parameters of the model were obtained by maximizing the log-likelihood function of the models using the bbmle package in R [16]. Their performance was then compared with new generalized Weibull distribution (NGW) [17] using goodness of fit measures such as Anderson–Darling statistic(AD), Cramer-von misses distance values (CM), Kolmogorov–Smirnov statistic(KS), Akaike information criteria (AIC), Bayesian information criteria (BIC) and Hannan Quinn information Criteria (HQIC). The smaller the value of the goodness of fit measures the better the fit to the data. The negative log-likelihood was also considered for the sake of comparison. The cdf and pdf of the NGW are respectively given by

$$F(x;\alpha,\eta,\theta,\varphi)=\left[1-e^{\left(-\alpha x-\eta x^{\theta}\right)}\right]^{\varphi}$$

and

$$f(x;\alpha,\eta,\theta,\varphi)=\varphi\left(\alpha+\eta\theta x^{\theta-1}\right)e^{\left(-\alpha x-\eta x^{\theta}\right)}\left[1-e^{\left(-\alpha x-\eta x^{\theta}\right)}\right]^{\varphi-1},x>0,\alpha>0,\eta>0,\theta>0,\varphi>0.$$

The first dataset used for the application, Data 1 in Table 2 consists of lifetimes of 50 components, given by [18] and the second dataset Data 2 in Table 3 represent the lifetime of a certain device given by [19] both found in [17].

A total time on test (TTT) transform plot of the dataset in Fig. 4 shows that both datasets have modified bathtub failure rate function.

6.1 First Application

All the parameters of OCL, OCW and NGW distributions were significant at 95%confidence level, however only $\hat{\lambda}$ of the OCB distribution was significant at 5% level of

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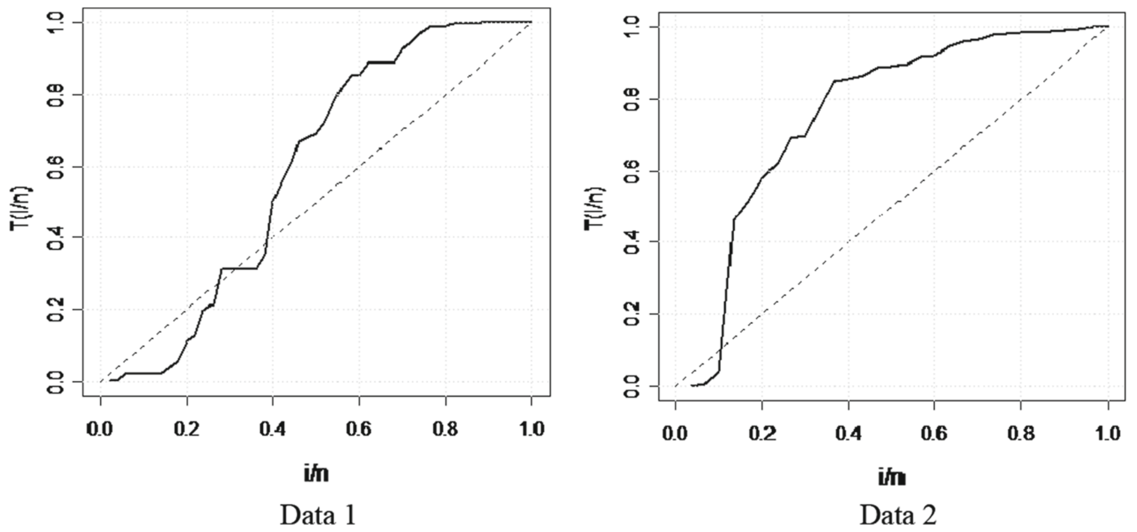

Fig. 4 TTT-transform plot for the datasets

Table 4 Maximum likelihood and standard error (in parenthesis) estimates of parameters

model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{k}$	$\hat{\eta}$	$\hat{\varphi}$
OCB		0.0275	0.1241	2.9023	3.0213			
		(0.0133)	(0.1756)	(3.909)	(1.9036)			
OCL		0.1199	0.3308	0.0074		6.8406		
		(0.0434)	(0.0828)	(0.003)		(2.2717)		
OCW	45.9988	0.3606	0.0310		5.0701			
	(4.1288)	(0.0725)	(0.0071)		(0.8432)			
NGW	0.0245			0.0407			3.3297	78.6862
	(0.0042)			(0.0138)			(0.1532)	(0.0019)

Table 5 Log-likelihood estimates and goodness of fit measures

Model	ℓ	AIC	BIC	CAIC	KS	AD	W
OCB	– 232.01	472.0250	479.6731	472.9139	0.1678	2.0008	0.3177
OCL	**– 225.31**	**458.6183**	**466.2664**	**459.5072**	**0.1453**	**1.4385**	**0.2030**
OCW	– 291.65	591.2914	598.9395	592.1803	0.1912	1.6060	0.2553
NGW	– 235.60	479.2089	486.8570	480.0978	0.1620	2.3681	0.3800

Bolded values means best based on the goodness of fit measures

significance. Table 4 shows the estimates of the maximum likelihood and their respective standard errors for Data1.

OCL distribution outperforms the rest of the models as it has the highest log-likelihood and the lowest values of all the goodness of fit measure and provides a comparatively reasonable fit as shown in Table 5 and Fig. 5 respectively.

The estimated parameter values of the OCL distribution are the maxima as shown by the profile likelihood plots in Fig. 6.

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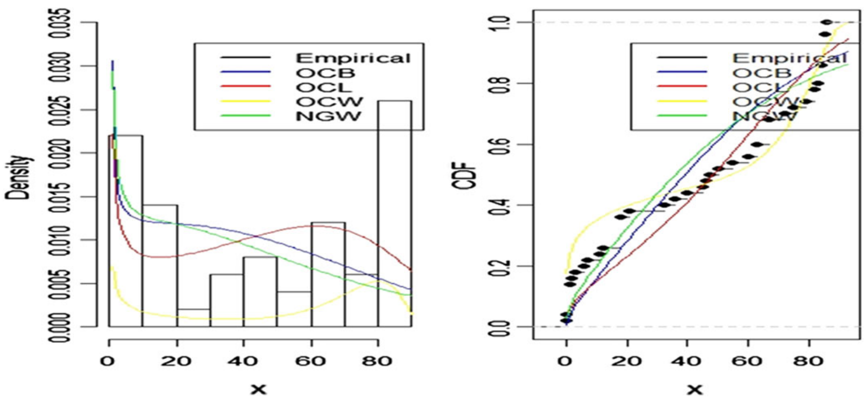

Fig. 5 Empirical and fitted density and cdf plots of fitted distributions for Data 1

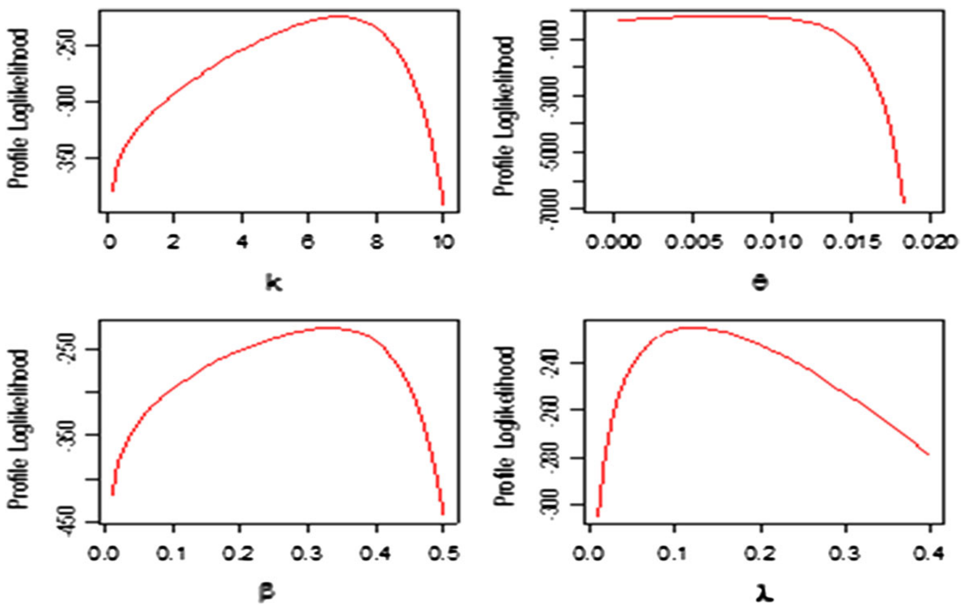

Fig. 6 Profile log-likelihood plot of OCL parameters

6.2 Second Application

The maximum likelihood and corresponding standard error estimates for Data2 is presented in Table 6. OCL distribution again provided a comparatively better fit for the dataset owing to the fact that it had the highest log-likelihood and the smallest values for all the goodness of fit measures used as shown in Table 7.

This is further confirmed by the plots of its empirical and fitted density and cdf plots in Fig. 7.

The estimated parameter values of the OCL distribution are the maxima as illustrated by the plot of profile likelihood in Fig. 8.

7 Conclusion

The flexibility of generalized models in modeling varying datasets remains a strong motivation for developing new families of distributions. The study developed a new

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Table 6 Maximum likelihood and standard error (in parenthesis) estimates of parameters

model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{k}$	$\hat{\eta}$	$\hat{\varphi}$
OCB		0.0054	0.0619	11.0047	0.4419			
		(0.0036)	(0.0042)	(0.0014)	(0.1512)			
OCL		0.0402	0.5094	0.0201		12.2327		
		(0.0259)	(0.2121)	(0.01)		(5.5816)		
OCW	4.7534	0.2015	0.0408		3.092			
	(1.2735)	(0.0685)	(0.0181)		(0.828)			
NMW	0.202			0.0131			3.2355	106.7165
	(0.03)			(0.0097)			(0.2269)	(0.0021)

Table 7 Log-likelihood estimates and goodness of fit measures of fitted distributions

model	ℓ	AIC	BIC	CAIC	KS	AD	W
OCB	− 75.97	159.9333	165.5381	161.5333	0.1414	0.9131	0.1343
OCL	**− 72.19**	**152.3888**	**157.9936**	**153.9888**	**0.0971**	**0.4823**	**0.0666**
OCW	− 116.5	241.0092	246.614	242.6092	0.0199	0.3095	0.0478
NMW	− 85.87	179.7422	185.347	181.3422	0.2219	2.5874	0.4194

Bolded values means best based on the goodness of fit measures

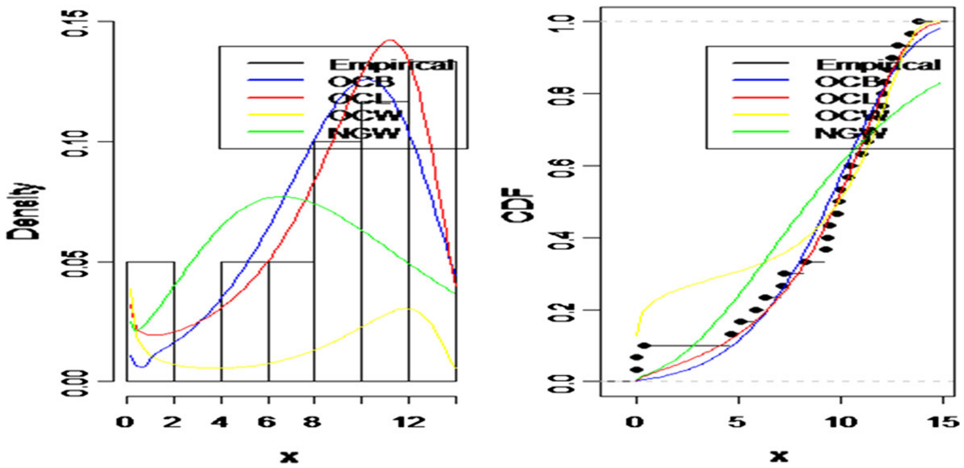

Fig. 7 Empirical and fitted density and cdf plots of fitted distributions for Data 2

family of distribution called the OCG family. Statistical properties such as the stochastic ordering, order statistics, moments, uncertainty measures and entropies of the new family are derived. Three special distributions of the new family were developed. Maximum likelihood estimates for the parameters of the special distributions were obtained. A demonstration of the application of the special distribution developed was carried out using two real datasets. A comparison of the results revealed that the OCL provided a better parametric fit to these datasets.

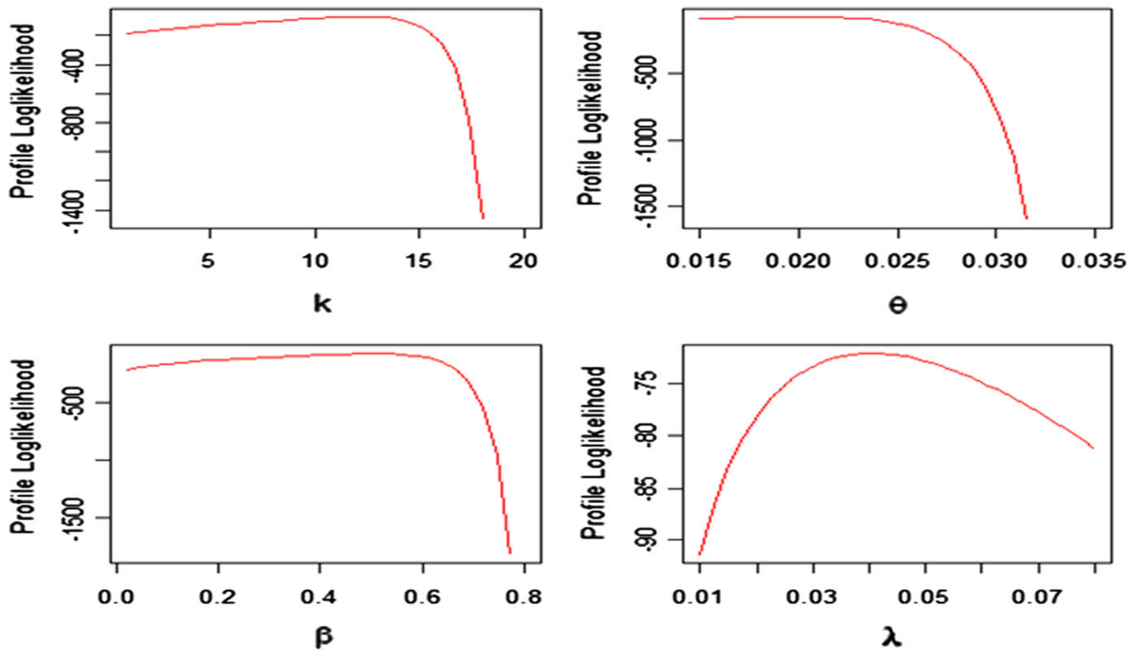

Fig. 8 Profile log-likelihood plot of OCL parameters

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Compliance with Ethical Standards

Conflict of interest The authors declare that there are no conflict of interest regarding the publication of this article.

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STATISTICS | RESEARCH ARTICLE

Chen-G class of distributions

Lea Anzagra[1]*, Solomon Sarpong[1] and Suleman Nasiru[1]

Abstract: The quest to generate distributions with more desirable and flexible properties for the modeling of data has led to an intense focus on the development of new families that are generalizations of existing distributions by researchers. A new family of distributions called the chen generated family is developed in this study. Its statistical properties such as the quantile, moments, incomplete moments, stochastic ordering and order statistics are derived by using the method of maximum likelihood, estimators for the parameters of the new family are developed. Three special distributions, Chen Burr III, Chen Kumaraswamy and Chen Weibull, are proposed from the new family, though it can generalize other distributions. A demonstration of the usefulness of the new family is performed using real dataset.

Subjects: advanced mathematics; applied mathematics; statistics & probability

Keywords: Chen; Weibull; distribution; moments; stochastic ordering; quantile
Jel: 62e15; 60e05

1. Introduction

The accuracy of parametric statistical inference and modeling of datasets largely depends on how well the probability distribution fits the given dataset once it has met all distributional assumptions. Several studies have been carried out on statistical distributions in the quest to generate distributions with more desirable and flexible properties that can model real-life datasets of varying shapes of density and failure rate functions. Currently, most studies are focused on developing new families that are generalizations of existing distributions to provide better fit to the modeling of data. These families of distributions are constructed by either compounding two or more distributions or adding one or more parameters to the baseline model. Many authors have extensively reviewed the various families of distributions (Hamedani, Yousof, Rasekhi, Alizadeh, & Najibi, 2018; Lee, Famoye, & Alzaatreh, 2013; Nasiru, 2018; Nasiru, Mwita, & Ngesa, 2018; Zubair, 2018).

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PUBLIC INTEREST STATEMENT

Modeling of natural phenomena such as earthquakes, rainfall, tsunami and so on mostly involves the use of statistical distributions. Since the accuracy of the results largely depends on how well the distribution fits the dataset, the study develops a new family of distributions which is to improve the flexibility of existing distributions.

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In this study a new class of distributions is developed and proposed using the T-X approach (Alzaatreh, Lee, & Famoye, 2013). The Chen generated (CG) family of distributions is obtained by compounding the two-parameter Chen distribution (Chen, 2000) and an arbitrary baseline cumulative distribution function (cdf) of a continuous random variable. The main motivation for developing this family is to improve the flexibility of the existing classical distributions, thus to enabling them to provide a better fit to real data sets than other candidate distributions with the same number of parameters and model different kinds of failure rate (monotonic and non-monotonic).

The remaining sections of the paper follow this order: the Chen generated (CG) family of distributions is defined in section 2. The mixture representation of the probability density function (pdf) is presented in section 3. Some statistical properties of the family of distributions are derived in section 4. The estimators for the parameters of the family are developed in section 5. Some special distributions from the CG family of distributions are proposed and discussed in section 6. Simulations to examine the properties of estimators of parameters of the special distributions are carried out in section 7. Real-life data set is used to demonstrate the application of the special distributions in section 8. Concluding remarks of the study are captured in section 9.

2. Chen generated a family of distributions

Let T be a Chen distributed continuous random variable, its cdf denoted by $F(t)$ is given by $F(t) = 1 - e^{\lambda\left(1-e^{t^{\beta}}\right)}, t > 0$ (Chen, 2000). Also, let $G(x)$and $g(x)$ be the respective cdf and pdf of an arbitrary continuous random variable X. The cdf of the CG family is defined as;

$$F(x) = \int_{0}^{G(x)} f(t)dt = A\left[1 - e^{\lambda\left(1-e^{G(x)^{\beta}}\right)}\right],\ x > 0,\ \lambda > 0,\ \beta > 0, \tag{1}$$

where $A = {}^{1}\!/_{1 - e^{\lambda(1-e)}}$ is a normalizing constant, λ and β are scale and shape parameters, respectively. The pdf $f(x)$ of the family is given by;

$$f(x) = A\lambda\beta g(x)G(x)^{\beta-1}e^{G(x)^{\beta}}e^{\lambda\left(1-e^{G(x)^{\beta}}\right)},\ x > 0,\ \lambda > 0,\ \beta > 0. \tag{2}$$

The survival function, $S(x)$of the CG family is;

$$S(x) = 1 - A\left[1 - e^{\lambda\left(1-e^{G(x)^{\beta}}\right)}\right],\ x > 0,\ \lambda > 0,\ \beta > 0. \tag{3}$$

The failure rate or hazard function, $h(x)$of the family is obtained as follows:

$$h(x) = \frac{A\lambda\beta g(x)G(x)^{\beta-1}e^{G(x)^{\beta}}e^{\lambda\left(1-e^{G(x)^{\beta}}\right)}}{1 - A\left[1 - e^{\lambda\left(1-e^{G(x)^{\beta}}\right)}\right]},\ x > 0,\ \lambda > 0,\ \beta > 0. \tag{4}$$

3. Mixture representation of distribution

Mixture representation plays a useful role in the derivation of the statistical properties of the new family of distributions. Hence, the mixture representation of the pdf of the CG family of distributions is derived in this section.

By applying Taylor series expansion, the pdf of the CG family in Equation (2) is expressed as

$$f(x) = A\lambda\beta e^{\lambda}g(x)G(x)^{\beta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}\lambda^{i}}{i!}\frac{(i+1)^{j}}{j!}G(x)^{\beta(j+1)-1}. \tag{5}$$

Equation (5) can be rewritten as;

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$$f(x) = A\lambda\beta e^{\lambda} g(x) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i \lambda^i}{i!} \frac{(i+1)^j}{j!} [1-(1-G(x))]^{\beta(j+1)-1}.$$

$f(x)$ is further expanded using the binomial series expansion $(1-z)^{a-1} = \sum_{k=0}^{\infty} (-1)^k \binom{a-1}{k} z^k$, $|z| < 1$ for any real non-integer $a > 0$ as follows:

$$f(x) = A\lambda\beta e^{\lambda} g(x) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (i+1)^j \lambda^i}{i!j!} \sum_{k=0}^{\infty} (-1)^k \binom{\beta(j+1)-1}{k} (1-G(x))^k.$$

Assuming a an integer in the binomial expansion,

$$f(x) = A\lambda\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \omega_{ijkl}\, g(x)(G(x))^l, \tag{6}$$

where

$$\omega_{ijkl} = \frac{(-1)^{i+k+l} (i+1)^j \lambda^i e^{\lambda}}{i!j!} \binom{\beta(j+1)-1}{k} \binom{k}{l}.$$

From Equation (6), the CG family's density is expressed as a product of the parameters and the sum of the product of the pdf and weighted power series of the baseline distribution function $G(x)$.

Alternatively, Equation (6) can further be written in terms of the exponentiated-G (expo-G) density function as

$$f(x) = A\lambda\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \omega^{*}_{ijkl} \pi_{l+1}(x), \tag{7}$$

where $\omega_{ijkl}{}^{*} = \frac{\omega_{ijkl}}{l+1}$ and $\pi_{l+1}(x) = (l+1)g(x)(G(x))^l$ is the expo-G density function with the power parameter $(l+1)$.

4. Statistical properties

This section discusses some of the statistical properties of the CG family of distributions. These include: quantile function, non-central moments, moments, generating functions, inequality measures, entropies, residual life, stochastic ordering and order statistics.

4.1. Quantile function

Proposition 1. The quantile function for CG family of distributions is given by

$$Q_G(u) = x_u = G^{-1}\left(\ln\left[1 - \frac{\ln(1 - {}^{u}/_{A})}{\lambda}\right]\right)^{\frac{1}{\beta}}, \; 0 < u < 1, \tag{8}$$

Proof. The quantile function $Q_G(u)$ of a random variable X is defined as $F(x_u) = P(X \le x_u) = u, u \in (0,1)$. Replacing x with x_u in Equation (1), equating $F(x_u)$ to u and making x_u the subject yields the quantile function. The median of the family is obtained by substituting $u = 0.5$ in Equation (8).

4.2. Moments, moment generating functions and incomplete moments

Moments are very essential in statistical analysis as they can be used to study important features (such as tendencies, variation, skewness, kurtosis and so on) of a distribution.

4.2.1. Non-central moments

Proposition 2. The r^{th} non-central moment of the CG family is given by

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$$\mu_r^{'} = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl}\tau_{(r,l)} \text{ ,} r = 1, 2, \ldots, \tag{9}$$

where $\tau_{(r,l)} = \int_{-\infty}^{\infty} x^r g(x)(G(x))^l dx$ is the probability weighted moment of the baseline distribution $G(x)$.

Proof. The r^{th} non-central moment is defined as $E(X^r) = \mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx$, thus using the mixture form of the density, the r^{th} non-central moment of the CG family is given by $\mu_r^{'} = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{\infty} x^r g(x)(G(x))^l dx$.

Alternatively, the r^{th} non-central moment of the CG family can be described in terms of the quantile function as follows;

Let $G(x) = u,\ x = G^{-1}(u) = Q_G(u),\ \frac{dG(x)}{dx} = \frac{du}{dx} = g(x)$ and $g(x)dx = du$. From Equation (9),

$$\mu_r^{'} = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_0^1 Q_G(u)^r u^l du. \tag{10}$$

4.2.2. Moment generating functions

Proposition 3. The moment generating function of the CG family is given by

$$M_X(t) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{k} \frac{(t)^r}{r!}\omega_{ijkl}\tau_{(r,l)}. \tag{11}$$

Proof. By definition, the moment generating function is given by $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$, expanding $M_X(t)$ using Taylor series, $M_X(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_{-\infty}^{\infty} x^r f(x)dx$.

But from Equation (9), $\int_{-\infty}^{\infty} x^r f(x)dx = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl}\tau_{(r,l)}$, hence the proof.

Alternatively, letting $G(x) = u$, the moment generating function can be expressed in terms of quantile functions as;

$$M_X(t) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_0^1 e^{tQ_G(u)} u^l du. \tag{12}$$

4.2.3. Incomplete moments

Proposition 4. The r^{th} incomplete moment of the CG family of distribution is given by

$$M_r(y) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} x^r g(x)(G(x))^l dx, r = 1, 2, \ldots. \tag{13}$$

Proof. The r^{th} incomplete moment is defined as $M_r(y) = \int_{-\infty}^{y} x^r f(x)dx$. Substituting the mixture representation of the density function into the definition of the r^{th} incomplete moments completes the proof.

Alternatively, letting $G(x) = u$, the incomplete moments can be expressed in terms of the quantile function as;

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$$M_r(y) = A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_0^{G(y)} Q_G(u)^r u^l du. \quad (14)$$

4.3. Inequality measures

Lorenz and Bonferroni curves are applied in so many fields such as econometrics, demography, reliability, medicine and insurance. They are generally used in studying inequality measures like income and poverty.

4.3.1. Lorenz curve

The Lorenz curve $L_F(y)$ for incomplete moments is defined as $L_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx$ for the CG family, it is given by;

$$L_F(y) = \frac{A\lambda\beta}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} xg(x)(G(x))^l dx. \quad (15)$$

Alternatively, letting $G(x) = u$, $L_F(y)$ can be expressed in terms of the quantile functions as;

$$L_F(y) = \frac{A\lambda\beta}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_0^{G(y)} Q_G(u)u^l du \quad (16)$$

4.3.2. Bonferroni curve

Bonferroni curve $B_F(y)$ is defined as $B_F(y) = \frac{L_F(y)}{F(y)}$, hence for the CG family it is given by;

$$B_F(y) = \frac{A\lambda\beta}{\mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} xg(x)(G(x))^l dx. \quad (17)$$

4.4. Mean residual life

The mean residual life of a component (which is the average survival time of the component after it has exceeded a specific timey) is defined as $E(X - y/X>y)$.

Proposition 5. The mean residual life of a CG random variable Y is given by

$$\overline{M}(y) = \frac{1}{1-F(y)}\left[\mu - A\lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \omega_{ijkl} \int_{-\infty}^{y} xg(x)(G(x))^l dx\right] - y. \quad (18)$$

Proof. The mean residual life is defined as $\overline{M}(y) = \frac{1}{1-F(y)}\left[\mu - \int_{-\infty}^{y} xf(x)dx\right] - y$. Substituting $f(x)$ in Equation (6) into $\overline{M}(y)$ gives the mean residual life.

4.5. Entropy

Entropy is a measure of variation or uncertainty of a random variable. Its application spans across probability theory, engineering and science in general.

4.5.1. Rényi's entropy

The Rényi's entropy (Rényi, 1961) for a random variable with pdf $f(x)$, is defined as;

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty} f^{\delta}(x)dx\right],\ \delta\neq 1, \delta>0$$

Proposition 5. Renyi's entropy for the CG random variable is given by;

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[(A\lambda\beta)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \varpi_{ijkl} \int_{-\infty}^{\infty} g(x)^{\delta}(G(x))^l dx\right],\ \delta\neq 1, \delta > 0, \quad (19)$$

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where

$$\varpi_{ijkl} = \frac{(-1)^{i+k+l}(\lambda\delta)^i}{i!}\frac{(i+\delta)^j}{j!}e^{\lambda\delta}\binom{\beta(j+\delta)-1}{k}\binom{k}{l}$$

Proof. From Equation (2), $f^{\delta}(x) = (A\lambda\beta)^{\delta} g(x)^{\delta} G(x)^{\delta\beta-1} e^{\delta G(x)^{\beta}} e^{\lambda\delta} e^{-\lambda\delta e^{G(x)^{\beta}}}$

Adopting similar concept for expanding the density, $f^{\delta}(x)$ becomes

$$f^{\delta}(x) = (A\lambda\beta)^{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\varpi_{ijkl}\, g(x)^{\delta}(G(x))^{l}$$

where $\varpi_{ijkl} = \frac{(-1)^{i+k+l}(\lambda\delta)^i}{i!}\frac{(i+\delta)^j}{j!}e^{\lambda\delta}\binom{\beta(j+\delta)-1}{k}\binom{k}{l}$. Substituting $f^{\delta}(x)$ into $I_R(\delta)$ completes the proof.

4.6. Stochastic ordering

Ordering mechanism in data can easily be shown using stochastic ordering. Let X and Y be random variables with cdfs $F_X(x)$ and $F_Y(x)$ respectively. X is less than Y in likelihood ratio order ($X \leq_{lr} Y$), if the function $f(x)/g(x)$ is decreasing for all x.

Proposition 6. Let $X\tilde{C}G(\lambda_1, \beta, \psi)$ and $Y\tilde{C}G(\lambda_2, \beta, \psi)$, where ψ is a $(p \times 1)$ vector of parameters associated with the baseline distribution. X is less than Y in likelihood ratio order ($X \leq_{lr} Y$) if $\lambda_2 < \lambda_1$.

Proof. The ratio of their pdfs is given by $\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_1-\lambda_2)(1-e^{G(x)^{\beta}})}$, which is a decreasing function if $\lambda_2 < \lambda_1$.

4.7. Order statistics

The pdf for the p^{th} order statistic $X_{p:n}$, of an independent identically distributed random sample $X_1, X_2, \ldots, X_n$ of size n,$f_{X_{p:n}}(x)$, is given by;

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!}[F(x)]^{p-1}[1-F(x)]^{n-p}f(x), p = 1, 2, \ldots, n.$$

Expanding $[F(x)]^{p-1}$ using binomial expansion, $[F(x)]^{p-1} = \sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}[1-F(x)]^{i}$. Substituting into the density of the p^{th} order statistic yields,

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}[S(x)]^{n-p+i}f(x)$$

where $[S(x)]^{n-p+i} = [1-F(x)]^{n-p+i}$.

Hence, the pdf for the p^{th} order statistic is given by;

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}A\lambda\beta g(x)G(x)^{\beta-1}e^{G(x)^{\beta}} \times e^{\lambda(n-p+i+1)\left(1-e^{G(x)^{\beta}}\right)} \tag{20}$$

Employing a similar concept of expanding the density of the CG family, a mixture representation of the pdf of the p^{th} order statistic is defined as;

$$f_{X_{p:n}}(x) = \frac{n!A\lambda\beta}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}D_{ijklm}g(x)G(x)^{m}, \tag{21}$$

where

$$D_{ijkl} = \frac{(-1)^{i+j+l+m}[\lambda(n-p+i+1)]^j}{i!}\frac{(j+1)^k}{k!}\binom{p-1}{i}\binom{\beta(k+1)-1}{l}\binom{l}{}e^{\lambda(n-p+i+1)}.$$

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4.7.1. Moments of order statistics

The r^{th} non-central moment of the p^{th} order statistic is given by $E(X^r_{p:n}) = \mu_r^{'(p:n)} = \int_{-\infty}^{\infty} x^r f_{X_{p:n}}(x)dx$. Substituting Equation (21) into $E(X^r_{p:n})$, the r^{th} non-central moment of the p^{th} order statistic of the CG random variable is given by,

$$E(X^r_{p:n}) = \frac{n!A\lambda\beta}{(p-1)!(n-p)!}\sum_{i=0}^{p-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l} D_{ijklm}\tau_{(r,m)} \tag{22}$$

where $\tau_{(r,m)} = \int_{-\infty}^{\infty} x^r g(x)G(x)^m dx$ is the probability weighted moment of the baseline distribution.

5. Parameter estimation

Maximum likelihood estimation method was used in estimating the parameters for the family of distribution for similar reasons as stated in Nasiru et al. (Nasiru et al., 2018). Given a random sample $x_1, x_2, \ldots, x_n$ of size n from the CG family of distributions, the total log-likelihood function is given by

$$\ell = n\log A\lambda\beta + \sum_{i=1}^{n}\log g(x_i;\psi) + (\beta-1)\sum_{i=1}^{n}\log G(x_i;\psi) + \sum_{i=1}^{n}G(x_i;\psi)^{\beta} + \lambda\sum_{i=1}^{n}\left(1-e^{G(x_i;\psi)^{\beta}}\right), \tag{23}$$

where ψ is a $(p\times 1)$ vector of parameters associated with the baseline distribution.

The parameters are then estimated by partially differentiating the total log-likelihood function with respect to the parameters of the CG family as follows.

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \frac{n(1-e)e^{\lambda(1-e)}}{1-e^{\lambda(1-e)}} + \sum_{i=1}^{n}(1-e^{G(x_i;\psi)^{\beta}}), \tag{24}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log G(x_i;\psi) + \sum_{i=1}^{n}G(x_i;\psi)^{\beta}\log G(x_i;\psi) - \lambda\sum_{i=1}^{n}G(x_i;\psi)^{\beta}e^{G(x_i;\psi)^{\beta}}\log G(x_i;\psi) \tag{25}$$

and

$$\frac{\partial\ell}{\partial\psi} = \sum_{i=1}^{n}\frac{g_k'(x_i;\psi)}{G(x_i;\psi)} + (\beta-1)\sum_{I=1}^{n}\frac{G_k'(x_i;\psi)}{G(x_i;\psi)} + \sum_{i=1}^{n}G_K'(x_i;\psi)G(x_i;\psi)^{\beta-1} - \lambda\beta\sum_{i=1}^{n}G_K'(x_i;\psi)G(x_i;\psi)^{\beta-1}e^{G(x_i;\psi)^{\beta}}, \tag{26}$$

where $g_K'(x_i;\psi) = \frac{\partial g(x_i;\psi)}{\partial\psi}$ and $G_K'(x_i;\psi) = \frac{\partial G(x_i;\psi)}{\partial\psi}$.

Equating the score functions to zero and numerically solving the system of equations using techniques such as the quasi Newton-Raphson method, gives the maximum likelihood estimates for the parameters. The interval estimates of the parameters are obtained by first finding the observed $(p\times p)$ information matrix given by $J(\vartheta) = \frac{\partial^2\ell}{\partial q\partial r}$ (for $q, r = \lambda, \beta, \psi$ and $\vartheta = (\lambda, \beta, \psi)^T$), whose elements can be numerically computed. Under the regularity conditions, as $n\to\infty$, $\hat{\vartheta}\sim N_p(0, J(\hat{\vartheta})^{-1})$, where $J(\hat{\vartheta})$ is the observed information matrix evaluated at $\hat{\vartheta}$. The approximate100$(1-\rho)\%$ confidence intervals (where ρ is the significance level) can be constructed using the asymptotic normal distribution.

6. Some special distributions

The CG family of distributions can be used to extend many distributions to create more flexibility in their applications. In this section some special distributions were developed.

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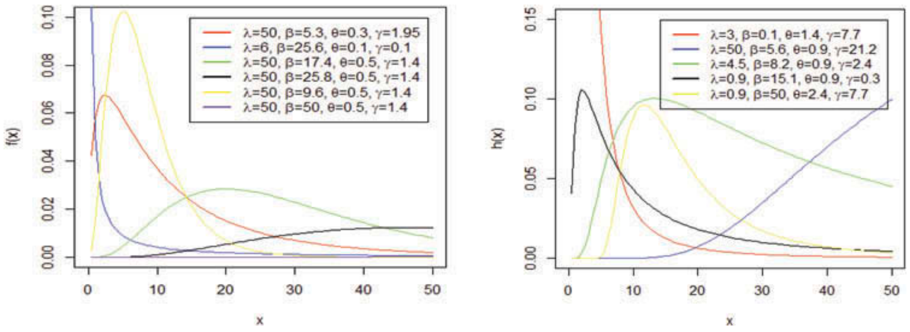

Figure 1. Plots of density and hazard rate functions of CB distribution.

6.1. Chen Burr III distribution

Suppose that the baseline distribution is Burr III (Burr, 1942), it's cdf and pdf are given by $G(x) = (1+x^{-\theta})^{-\gamma}$ and $g(x) = \gamma\theta x^{-\theta-1}(1+x^{-\theta})^{-\gamma-1}$, $x > 0,\ \theta > 0,\ \gamma > 0$ respectively. The cdf of Chen Burr III (CB) is given by

$$F(x) = A\left[1 - \exp\left(\lambda\left(1 - e^{(1+x^{-\theta})^{-\gamma\beta}}\right)\right)\right],\ x > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0,\ \lambda > 0. \tag{27}$$

Its corresponding density and hazard functions are, respectively

$$f(x) = A\lambda\beta\gamma\theta(x)^{-\theta-1}(1+x^{-\theta})^{-\gamma\beta-1}\exp\left[(1+x^{-\theta})^{-\gamma\beta} + \lambda\left(1 - e^{(1+x^{-\theta})^{-\gamma\beta}}\right)\right],\ x > 0 \tag{28}$$

and

$$h(x) = \frac{A\lambda\beta\gamma\theta(x)^{-\theta-1}(1+x^{-\theta})^{-\gamma\beta-1}\exp\left[(1+x^{-\theta})^{-\gamma\beta} + \lambda\left(1 - e^{(1+x^{-\theta})^{-\gamma\beta}}\right)\right]}{1 - A\left[1 - \exp\lambda\left(1 - e^{(1+x^{-\theta})^{-\gamma\beta}}\right)\right]},\ x > 0. \tag{29}$$

Plots of the density and hazard rate functions of the CB distribution are displayed in Figure 1. The density plot exhibit varying shapes including unimodal with different degrees of kurtosis, right skewed and reversed J shapes. The hazard rate function for some selected values exhibited upside down bathtub, decreasing and increasing failure rates.

The CB distribution's quantile function $Q_G(u)$ is given by;

$$Q_G(u) = x_u = \left[\left(\log\left(1 - \left(\frac{\log(1 - {}^{u}\!/_{A})}{\lambda}\right)\right)\right)^{-\frac{1}{\gamma\beta}} - 1\right]^{-\frac{1}{\theta}}.$$

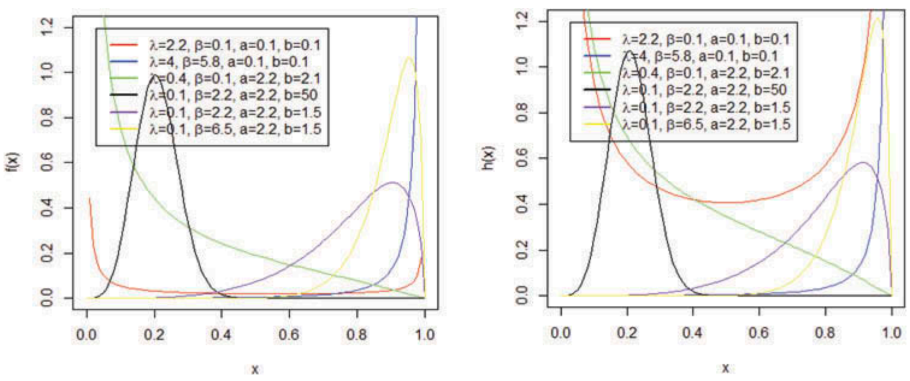

Figure 2. Plots of the density and hazard rate function of CK distribution.

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6.2. Chen Kumaraswamy distribution

The Chen Kumaraswamy (CK) distribution uses the Kumaraswamy distribution (Kumaraswamy, 1980) with pdf and cdf respectively given by $G(x) = 1-(1-x^a)^b$ and $g(x) = abx^{a-1}(1-x^a)^{b-1}$, $0<x<1,\ a>0,\ b>0$ as the baseline distribution. The cdf of CK distribution is given by

$$F(x) = A\left[1-\exp\lambda\left[1-e^{\left[1-(1-x^a)^b\right]^\beta}\right]\right]\quad x>0,\ a>0,\ b>0,\ \beta>0,\ \lambda>0, \tag{30}$$

with its corresponding density and hazard rate functions, respectively, given by

$$f(x) = A\lambda\beta abx^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\beta-1}\exp\left[\left(1-(1-x^a)^b\right)^\beta+\lambda\left(1-e^{\left(1-(1-x^a)^b\right)^\beta}\right)\right],\ x>0 \tag{31}$$

and

$$h(x) = \frac{A\lambda\beta abx^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\beta-1}\exp\left[\left(1-(1-x^a)^b\right)^\beta+\lambda\left(1-e^{\left(1-(1-x^a)^b\right)^\beta}\right)\right]}{1-\left[1-\exp\lambda\left[1-e^{\left[1-(1-x^a)^b\right]^\beta}\right]\right]},\ x>0. \tag{32}$$

Plots of the density and hazard rate functions of the CK distribution are displayed in Figure 2. The plot of the density shows shapes such as; the reversed J, left skewed, right skewed and unimodal shapes among others. The hazard rate plot for some selected values exhibits increasing and decreasing failure rates, unimodal and bathtub shapes.

The quantile function $Q_G(u)$ is obtained as.

$$Q_G(u) = x_u = \left[1-\left(1-\left(\log\left(1-\left(\frac{\log(1-u/A)}{\lambda}\right)\right)\right)^{\frac{1}{\beta}}\right)^{\frac{1}{b}}\right]^{\frac{1}{a}}$$

6.3. Chen Weibull distribution

Chen Weibull (CW) distribution is obtained using Weibull distribution (Weibull, 1951) with cdf and pdf, respectively, given by $G(x) = 1-e^{-\left(\frac{x}{\alpha}\right)^\gamma}$ and $g(x) = \left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}e^{-\left(\frac{x}{\alpha}\right)^\gamma}$ as baseline distribution. The cdf and pdf of the CW distribution are, respectively, given by

$$F(x) = A\left[1-\exp\lambda\left[1-e^{\left(1-e^{-\left(\frac{x}{\alpha}\right)^\gamma}\right)^\beta}\right]\right],\ x>0,\ \alpha>0,\ \beta>0,\ \lambda>0,\ \gamma>0 \tag{33}$$

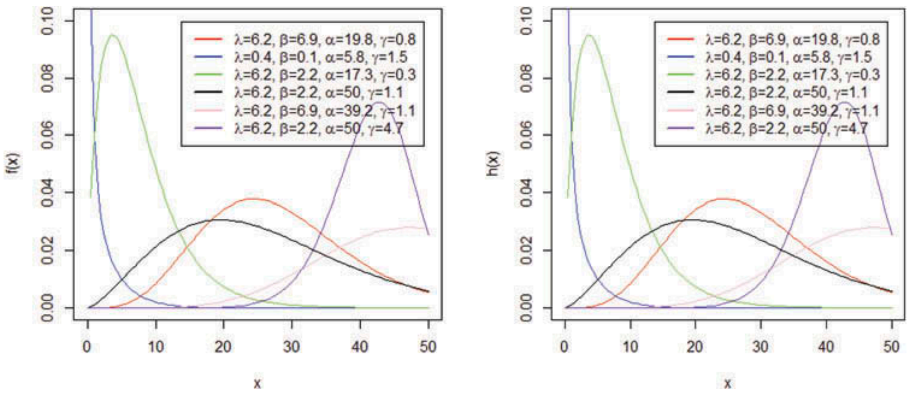

Figure 3. Plots of density and hazard rate function of CW distribution.

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Table 1. Monte Carlo simulation results

n	Parameter	I			II		
		AB	RMSE	CP	AB	RMSE	CP
50	λ	−0.5854	1.0708	0.9987	0.4737	0.9182	0.9913
	β	3.8663	56.4303	0.9977	2.6341	5.2212	0.9990
	α	−0.1005	0.1836	0.9977	0.6564	1.4488	0.9180
	γ	−0.0171	2.4805	0.9327	0.0876	0.3030	0.9920
150	λ	−0.2607	1.1688	0.9793	0.5179	0.9948	0.9873
	β	0.6373	1.2615	0.9945	1.6927	2.3435	0.9990
	α	−0.0534	0.1321	0.9867	0.6499	1.2680	0.9600
	γ	−0.1023	1.7291	0.9360	0.0652	0.1958	0.9927
300	λ	−0.1324	1.2618	0.9607	0.5254	1.0150	0.9793
	β	0.4988	0.9901	0.9973	1.5134	1.8010	0.9067
	α	−0.0396	0.1125	0.9853	0.5978	1.1114	0.9727
	γ	−0.2452	1.2307	0.9393	0.0484	0.1303	0.9913
600	λ	−0.0231	1.1910	0.9592	0.4924	1.0072	0.9580
	β	0.3936	0.5929	0.9900	1.4374	1.5874	0.7500
	α	−0.0240	0.0950	0.9633	0.5487	1.0468	0.9793
	γ	−0.2420	0.9657	0.9433	0.0405	0.1034	0.9827
1000	λ	0.0428	1.1763	0.9367	0.4089	0.8572	0.9393
	β	0.3599	0.5053	0.9640	1.3880	1.4766	0.6780
	α	−0.0173	0.0856	0.9407	0.4867	0.9565	0.9747
	γ	−0.2526	0.8181	0.9367	0.0402	0.0934	0.9513

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Table 2. Fatigue time of 101 6061-T6 aluminum coupons

10	33	44	56	59	72	74	77	92	93	96	100
100	102	105	107	107	108	108	108	109	112	113	115
116	120	121	122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	183	195	196
197	202	213	215	216	222	230	231	240	245	251	253
254	255	278	293	327	342	347	361	402	432	458	555

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Table 3. Survival times of guinea pigs injected with different amounts of tubercle bacilli

70	90	96	97	99	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121
123	124	124	124	124	124	128	128	129	129	130	130	130	131
131	131	131	131	132	132	132	133	134	134	134	134	136	136
137	138	138	138	139	139	141	141	142	142	142	142	142	142
142	142	144	144	145	146	148	148	149	151	151	152	155	156
155	156	157	157	157	157	158	159	162	163	163	164	166	166
168	170	174	201	212									

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and

$$f(x) = A\lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta-1}\exp\left[\lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta}-\left(\frac{x}{\alpha}\right)^{\gamma}+\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)\right],\ x>0. \tag{34}$$

The hazard rate function is given by

$$h(x) = \frac{A\lambda\beta\left(\frac{\gamma}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\gamma-1}\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta-1}\exp\left[\lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta}-\left(\frac{x}{\alpha}\right)^{\gamma}+\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)\right]}{1-A\left[1-\exp\lambda\left[1-e^{\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\gamma}}\right)^{\beta}}\right]\right]},\ x>0. \tag{35}$$

The CW distribution's plots of its density exhibit; right skewed, left skewed, unimodal and reversed J shapes among others as shown in Figure 3. The hazard rate plot of the CW distribution for some selected values exhibits varying shapes such as increasing and decreasing failure rates, right and left skewed unimodal shapes and upside down bathtub shape.

The quantile function $Q_G(u)$ of the CW distribution is given by

$$Q_G(u) = x_u = \alpha\left(-\log\left(1-\left(\frac{\log\left(1-{}^{u}/_{A}\right)}{\lambda}\right)^{\frac{1}{\beta}}\right)\right)^{\frac{1}{\gamma}}.$$

7. Simulations

Monte Carlo simulations were performed in this section to investigate the behavior of the maximum likelihood estimators of the parameters. For illustration purposes, the simulation experiments were undertaken using the Chen Weibull distribution. The experiments were replicated for $N = 1500$ times using sample size $n = 50, 150, 300, 600, 1000$ and parameter values $\mathbf{I}: \lambda = 1.9, \beta = 0.9, \alpha = 0.8, \gamma = 4.8$ and $\mathbf{II}: \lambda = 0.5, \beta = 0.5, \alpha = 0.5, \gamma = 0.5$. The average bias (AB), root-mean-square error (RMSE) and coverage probability (CP) of the 95% confidence intervals for the estimators of the parameters were estimated. From Table 1, the ABs and RMSEs for the estimators generally decrease to zero as the sample size increases. This implies that as the sample size increases the accuracy and consistency of the maximum likelihood estimators are achieved. Also, the CPs for most of the estimators are quite close to the nominal value of 0.95. Thus, we can say that the maximum likelihood technique works very well to estimate the parameters of the Chen Weibull distribution.

8. Applications

In this section the performance of the CW distribution in providing good parametric fits to real-life datasets is demonstrated. Its goodness of fit measures are compared with competing models such as; exponentiated Chen (EC) (Chaubey & Zhang, 2015), extended Weibull (EW) (Xie, Tang, & Goh, 2002) and

Figure 4. TTT-transform plots for the datasets.

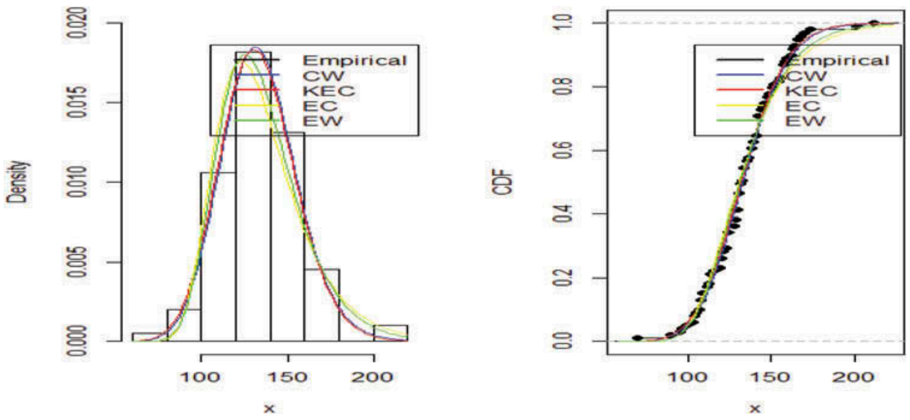

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Table 4. Maximum likelihood estimates and standard errors of parameters in brackets

Application	Model	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$
Data1	CW	6.775	35.209	49.29	1.019			
		(4.592)	(2.817)	(2.575)	(0.167)			
	KEC		0.324	0.021		4.473	4.336	2.286
			(0.014)	(0.005)		(0.676)	(0.698)	(1.557)
	EC	0.289	0.245	1236.1				
		(0.039)	(0.008)	(4.15×10^{-6})				
	EW	0.020	55.14	1.493				
		(0.001)	(7.27×10^{-4})	(0.106)				
Data 2	CW	19.366	15.742	30.945	0.31			
		(36.009)	(1.242)	(12.491)	(0.098)			
	KE-Chen		0.116	0.449		149.569	0.192	0.589
			(0.038)	(0.412)		(2.87×10^{-4})	(0.234)	(0.006)
	E-Chen	0.865	0.138	163.36				
		(0.077)	(0.008)	(1.79×10^{-4})				
	EW	4.626		242.255	0.272			
		(2.378)		(0.094)	(0.022)			

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Table 5. Goodness-of-fit statistics and information criteria

Application	Model	KS	CM	AD	AIC	BIC	CAIC	HQIC
Data1	CW	0.064	0.046	0.299	901.984	912.365	902.41	906.184
	KEC	0.061	0.047	0.323	904.436	917.411	905.081	909.686
	EW	0.096	0.134	0.783	907.241	915.026	907.494	910.391
	EC	0.11	0.194	1.127	913.19	920.975	913.442	916.34
Data2	CW	0.092	0.09	0.564	859.704	868.81	860.301	863.329
	KEC	0.093	0.094	0.581	861.604	872.987	862.513	866.136
	EW	0.145	0.232	1.603	877.894	884.724	878.247	880.613
	EC	0.124	0.153	1.089	869.634	876.464	869.987	872.354

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Figure 5. Empirical and fitted density and cdf plots of data1.

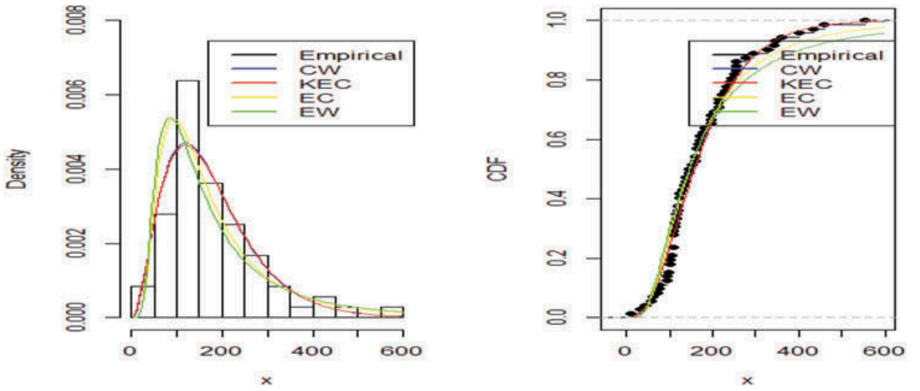
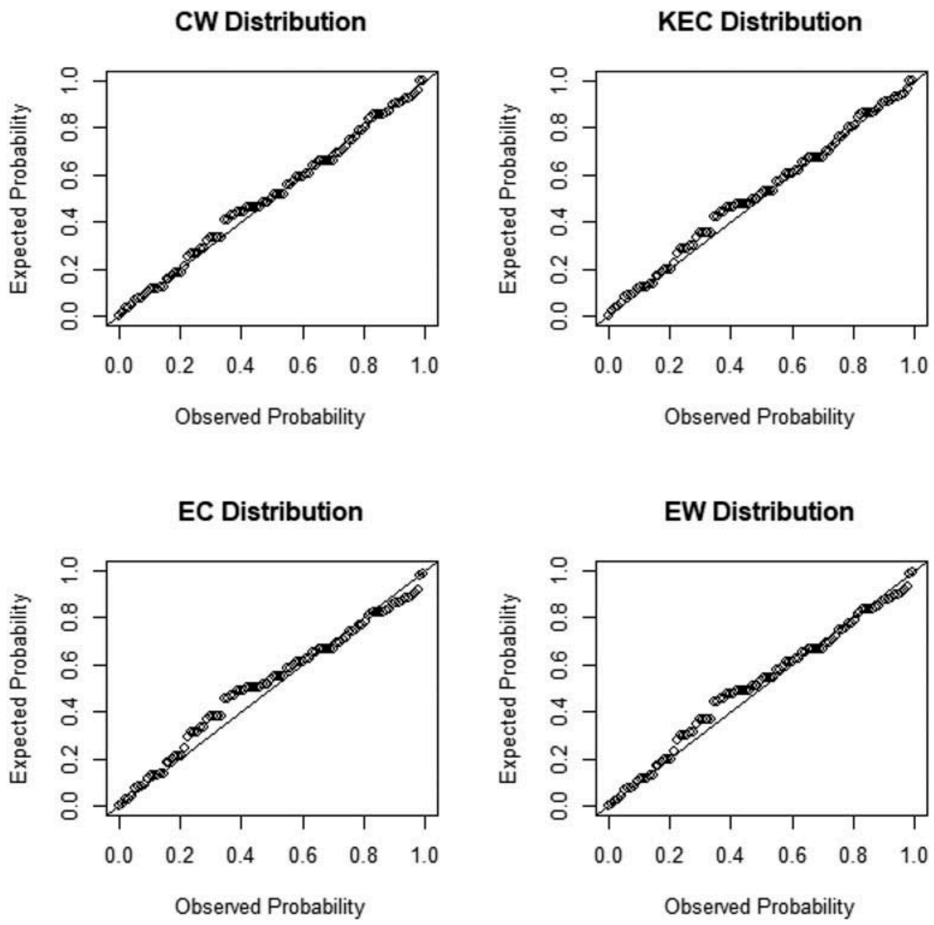


Figure 6. Empirical and fitted density and cdf plots of data2.



Kumaraswamy exponentiated Chen (KEC) (Khan, King, & Hudson, 2018) distributions. The information criteria and goodness of fit measures used are; Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent Akaike information criteria (CAIC), HQ information Criteria (HQIC), Kolmogorov–

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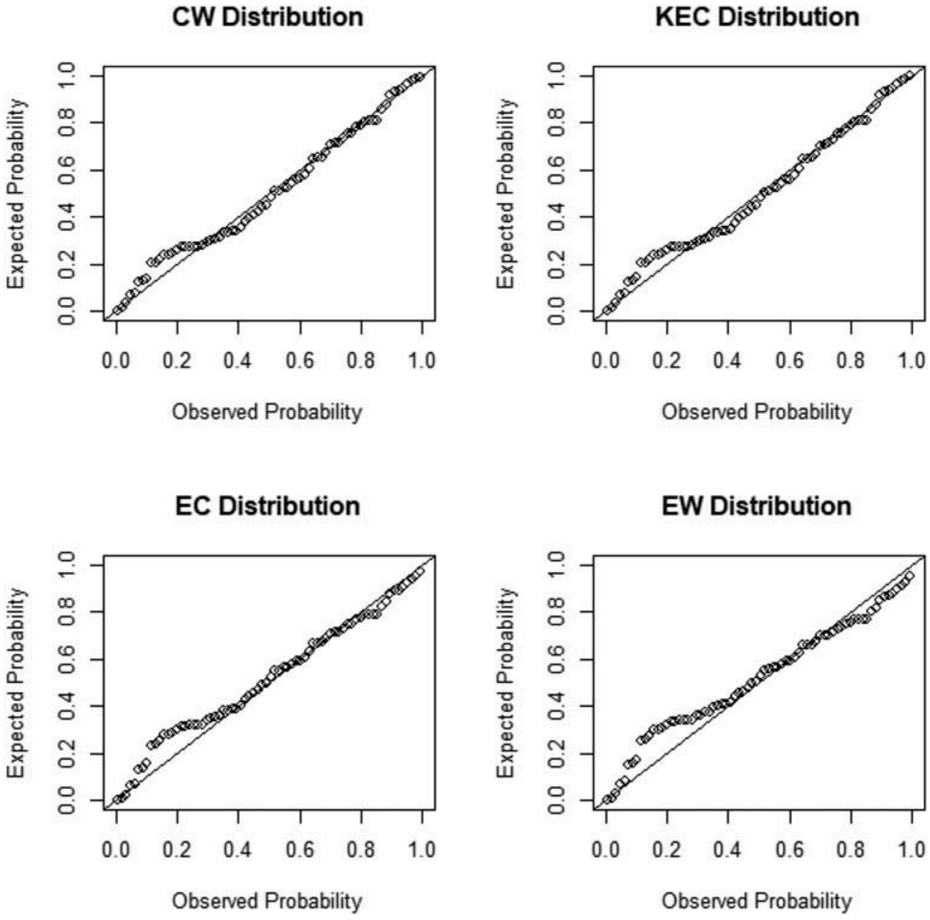
Figure 7. P-P plots of fitted distributions for data1.



Smirnov statistic(KS),Cramer-von misses distance values (CM) and Anderson Darling statistic (AD). In obtaining the maximum likelihood estimates for the parameters, the log-likelihood function of the models were maximized using the bbmle package's subroutine mle2 in R (Bolker, 2014). The maximum likelihood estimates with the largest maxima were chosen after using a wide range of initial values.

For illustration, the first dataset (data1) consists of the fatigue times of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second found in Birnbaum & Saunders (Birnbaum & Saunders, 1969), whilst the second dataset (data2) represents survival times of guinea pigs injected with different amounts of tubercle bacilli studied by Bjerkedal (Bjerkedal, 1960). These datasets are given in Tables 2 and 3.

A preliminary exploration of the datasets on the shapes of the hazard rate functions showed that data1 has an increasing hazard rate function whilst data two have a unimodal hazard rate function as shown in Figure 4.

The maximum likelihood estimates and the corresponding standard errors of the parameters of the fitted distributions for both datasets and their goodness of fit measures are displayed in Tables 4 and 5 respectively. The parameters of all the distributions were significant at 5% significance level, with the exception of CW and KEC distributions which had only one of their parameters ($\lambda$and$b$respectively) significant at 15% significance level.

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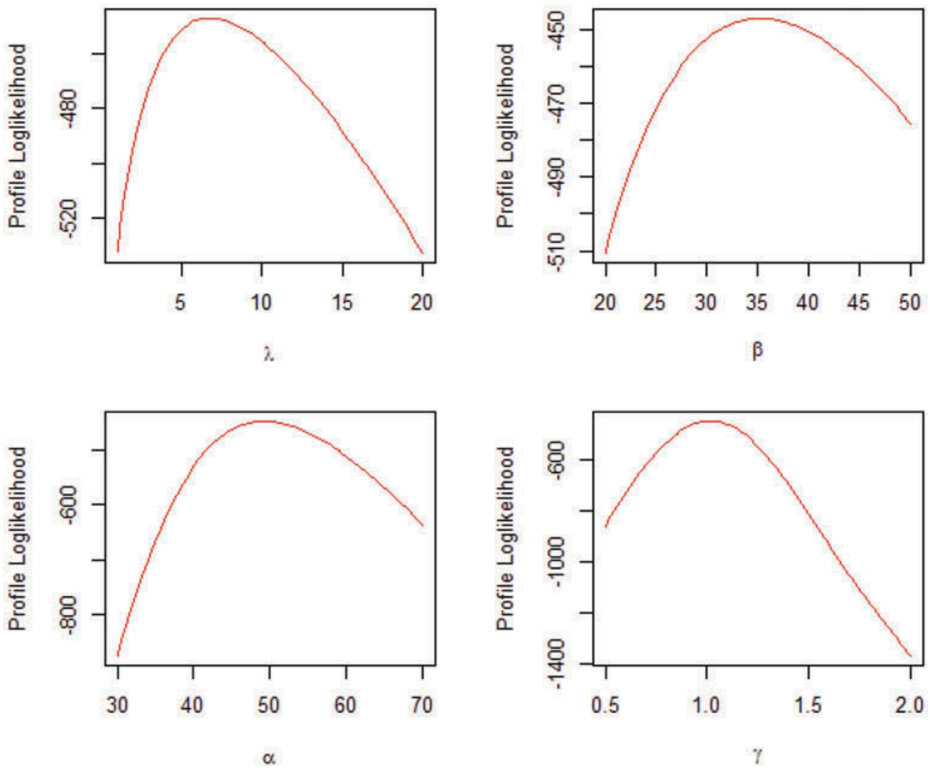
Figure 8. P-P plots of fitted distributions for data2.



Compared to the competing models, the CW distribution with its four parameters provides a better fit for the datasets as it has the smallest value for all the goodness of fit measures used as shown in Table 5.

This is further confirmed by the plots of densities and cdfs of the empirical and fitted distributions as shown in Figures 4 and 5. From the fitted plot, it is observed that the CW provides a reasonable fit to the density.

The P-P plots also indicates the CW distribution provides a better fit for both datasets in comparison with KEC, EC and EW distributions as shown in Figures 6 and 7.

The profile likelihoods of the estimated parameters of the CW distribution for the datasets are shown in Figures 8 and 9. From the plots, it is observed that the estimated values for the parameters are the maxima.

9. Conclusion

The focus of most researchers is geared towards developing new families of distributions for generalizing existing distributions to provide better fit for the modeling of life data. A new family of distribution called the CG family is developed and studied. Its statistical properties such as the quantile, moments, incomplete moments, generating function, entropies, stochastic ordering and order statistics are derived. Estimators for the parameters of the new family were developed using the method of maximum likelihood. A demonstration of the application of the special distribution developed from the family was carried out using two-real datasets. A comparison of the results with that of other existing distributions showed that the special distribution developed from the CG family provide a better parametric fit to these datasets.

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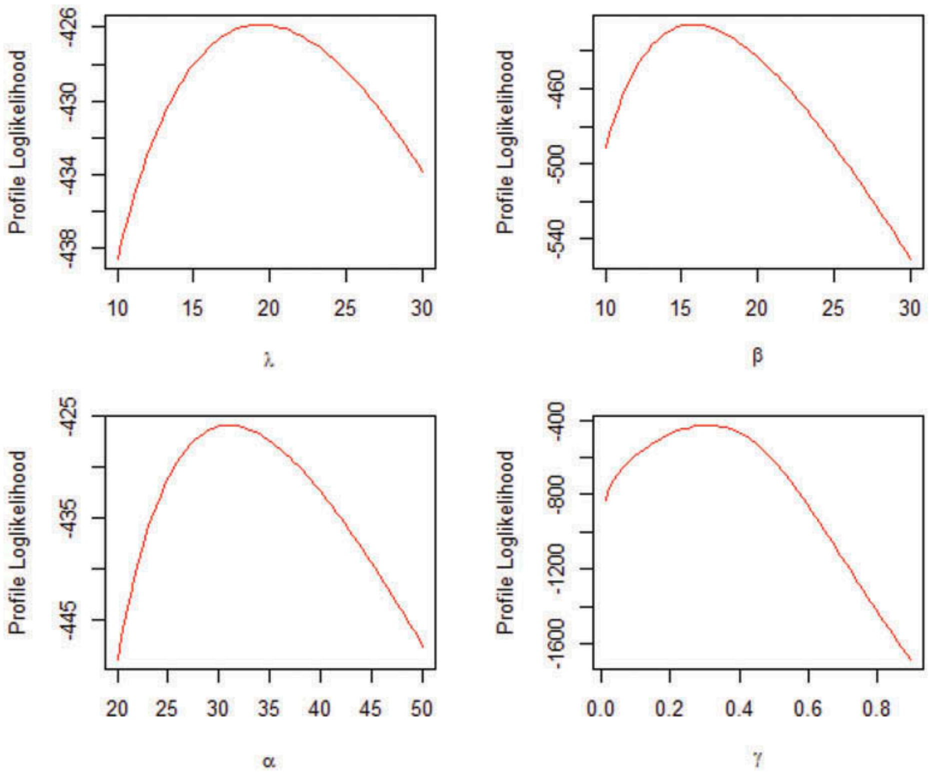
Figure 9. Profile log-likelihood plot of CW parameters for data1.



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